

Question 1

- (a) Sum the joint PMF over k to obtain the marginal PMF:

$$P_J(j) = \sum_k P_{J,K}(j, k) = \begin{cases} 0.6 & j = -2, \\ 0.4 & j = -1, \\ 0 & \text{otherwise.} \end{cases}$$

The discrete MGF is therefore

$$\phi_J(s) = \mathbb{E}[e^{sJ}] = \sum_j e^{sj} P_J(j) = 0.6e^{-2s} + 0.4e^{-s}.$$

- (b) Sum over j :

$$P_K(k) = \sum_j P_{J,K}(j, k) = \begin{cases} 0.56 & k = -1, \\ 0.24 & k = 0, \\ 0.20 & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\phi_K(s) = \sum_k e^{sk} P_K(k) = 0.56e^{-s} + 0.24 + 0.20e^s.$$

- (c) For each m , add the joint probabilities for which $j + k = m$:

$$P_M(m) = \sum_{j+k=m} P_{J,K}(j, k) = \begin{cases} 0.35 & m = -3, \\ 0.15 + 0.21 = 0.36 & m = -2, \\ 0.10 + 0.09 = 0.19 & m = -1, \\ 0.10 & m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The probabilities sum to one.

- (d) The product rule for MGFs of sums requires independence. Here,

$$P_{J,K}(-2, -1) = 0.35 \neq P_J(-2)P_K(-1) = 0.6(0.56) = 0.336.$$

Thus independence cannot justify a product of the marginal MGFs. Use the PMF in (c):

$$\begin{aligned} \phi_M(s) &= \sum_m e^{sm} P_M(m) = 0.35e^{-3s} + 0.36e^{-2s} + 0.19e^{-s} + 0.10, \\ \phi'_M(s) &= -1.05e^{-3s} - 0.72e^{-2s} - 0.19e^{-s}, \\ \phi''_M(s) &= 3.15e^{-3s} + 1.44e^{-2s} + 0.19e^{-s}. \end{aligned}$$

The second raw moment is

$$\boxed{\mathbb{E}[M^2] = \phi''_M(0) = 3.15 + 1.44 + 0.19 = 4.78.}$$

Question 2

- (a) For a Poisson random variable with parameter α ,

$$\phi_K(s) = \exp\{\alpha(e^s - 1)\}.$$

Substituting $\alpha = 2$ gives $\phi_{K_i}(s) = \exp\{2(e^s - 1)\}$, for all real s .

- (b) The K_j are independent, so

$$\phi_{R_i}(s) = \prod_{j=1}^i \phi_{K_j}(s) = [\exp\{2(e^s - 1)\}]^i = \exp\{2i(e^s - 1)\}.$$

- (c) This is the MGF of a Poisson random variable with parameter $2i$. MGFs that exist in a neighbourhood of zero uniquely determine the distribution, so

$$R_i \sim \text{Poisson}(2i), \quad P_{R_i}(r) = \begin{cases} (2i)^r e^{-2i} / r! & r = 0, 1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

- (d) A Poisson variable has mean and variance equal to its parameter:

$$\boxed{E[R_i] = 2i, \quad \text{Var}[R_i] = 2i.}$$

Question 3

- (a) The distributions of X_1 , X_2 and X_3 are unknown; only their means, variances and independence are given. Because no further distributional information is available, we use the central limit theorem (CLT) to construct a Gaussian approximation for the sum. Linearity gives its exact mean:

$$\mu_X = \sum_{i=1}^3 E[X_i] = 2.5 - 0.6 + 1.2 = 3.1.$$

Independence makes the cross-covariances zero, so its exact variance is

$$\sigma_X^2 = \sum_{i=1}^3 \text{Var}[X_i] = 1.8 + 2.4 + 0.9 = 5.1.$$

Let G be the requested Gaussian model with these moments. In the module's notation, a Gaussian distribution is written with its mean and standard deviation as the two parameters. Thus

$$G \sim \text{Gaussian}(\mu_X = 3.1, \sigma_X = \sqrt{5.1}), \quad f_G(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(x - \mu_X)^2}{2\sigma_X^2}\right) = \frac{\exp[-(x - 3.1)^2/10.2]}{\sqrt{10.2\pi}}, \quad x \in \mathbb{R}.$$

Accuracy is not guaranteed. The given means, variances and independence determine the first two moments of X , but not its distribution. Thus this is an information-limited, CLT-based approximation rather than an exact consequence of the supplied data. With only three arbitrary, non-identically distributed variables, the CLT does not guarantee a good Gaussian approximation; this is a prescribed model with matching moments.

- (b) Apply the Gaussian model, keeping the approximation distinct from an exact equality:

$$\begin{aligned} P(X \geq 5.36) &\approx P(G \geq 5.36) = 1 - \Phi\left(\frac{5.36 - \mu_X}{\sigma_X}\right) \\ &= 1 - \Phi\left(\frac{5.36 - 3.1}{\sqrt{5.1}}\right) = 1 - \Phi(1.0007448) \approx \boxed{0.1585}. \end{aligned}$$

A standard-normal table rounded to $z = 1.00$ gives $1 - 0.8413 = 0.1587$, which is also acceptable at that precision.

Question 4

For an exponential variable, $E[X] = 1/\lambda$ and $\text{Var}[X] = 1/\lambda^2$. Thus $\lambda = 1/8$, $\mu_X = 8$ and $\sigma_X^2 = 64$.

(a) For an iid sample mean,

$$\text{Var}[M_n(X)] = \frac{\text{Var}[X]}{n} \implies \boxed{\text{Var}[M_{16}(X)] = \frac{64}{16} = 4.}$$

(b) The exponential survival function is $P(X > x) = e^{-\lambda x}$ for $x \geq 0$. Hence

$$\boxed{P(X_1 > 10) = e^{-(1/8)10} = e^{-1.25} \approx 0.2865.}$$

(c) The sample mean has exact mean 8 and variance 4. Its CLT approximation is Gaussian with standard deviation 2:

$$\begin{aligned} P(M_{16}(X) > 10) &\approx 1 - \Phi\left(\frac{10 - \mu_X}{\sigma_X/\sqrt{16}}\right) \\ &= 1 - \Phi\left(\frac{10 - 8}{8/4}\right) = 1 - \Phi(1) \approx \boxed{0.1587}. \end{aligned}$$

Part (b) concerns one observation, while (c) concerns the average of 16 observations. Averaging reduces the variance from 64 to 4. The probability in (c) is a CLT approximation, not an exact exponential tail.

Question 5

(a) By linearity of expectation,

$$E[Y_n] = E[X_{2n-1} - X_{2n}] = E[X_{2n-1}] - E[X_{2n}] = E[X] - E[X] = \boxed{0}.$$

(b) For a difference,

$$\text{Var}[A - B] = \text{Var}[A] + \text{Var}[B] - 2 \text{Cov}[A, B].$$

The two samples are independent, so their covariance is zero:

$$\boxed{\text{Var}[Y_n] = \text{Var}[X] + \text{Var}[X] = 2 \text{Var}[X].}$$

(c) Different Y_n use disjoint pairs of independent samples. They therefore form an iid sequence. For its sample mean,

$$\begin{aligned} E[M_n(Y)] &= \frac{1}{n} \sum_{j=1}^n E[Y_j] = \boxed{0}, \\ \text{Var}[M_n(Y)] &= \frac{\text{Var}[Y_1]}{n} = \boxed{\frac{2 \text{Var}[X]}{n}}. \end{aligned}$$

Question 6

Each independent roll of the pair succeeds with probability $p = (1/6)^2 = 1/36$. The trial number of the third success has a Pascal distribution with $k = 3$:

$$E[R] = \frac{k}{p} = \frac{3}{1/36} = 108, \quad \text{Var}[R] = \frac{k(1-p)}{p^2} = \frac{3(35/36)}{(1/36)^2} = 3780.$$

- (a) Markov's inequality for nonnegative R gives

$$P(R \geq a) \leq \frac{E[R]}{a} \implies \boxed{P(R \geq 250) \leq \frac{108}{250} = 0.432.}$$

- (b) Chebyshev's inequality gives

$$P(|R - E[R]| \geq c) \leq \frac{\text{Var}[R]}{c^2}.$$

Here $250 - 108 = 142$, hence

$$P(R \geq 250) \leq P(|R - 108| \geq 142) \leq \frac{3780}{142^2} \approx \boxed{0.1875}.$$

In this example the first inequality is actually an equality: the extra lower-tail event would be $R \leq -34$, which is impossible.

- (c) The event $R \geq 250$ means that the first 249 rolls contain at most two successes. Let B count those successes; then $B \sim \text{Binomial}(249, 1/36)$. Only three terms are needed:

$$\begin{aligned} P(R \geq 250) &= P(B \leq 2) = \sum_{j=0}^2 \binom{249}{j} p^j (1-p)^{249-j} \\ &= \left(\frac{35}{36}\right)^{249} + 249 \left(\frac{1}{36}\right) \left(\frac{35}{36}\right)^{248} \\ &\quad + \binom{249}{2} \left(\frac{1}{36}\right)^2 \left(\frac{35}{36}\right)^{247} \\ &\approx \boxed{0.0299443}. \end{aligned}$$

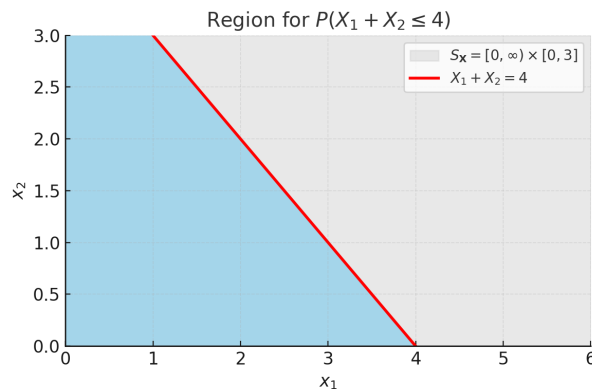
The finite expression is exact; the decimal is rounded.

Both bounds exceed the exact probability, as required. The Chebyshev bound is tighter here.

Question 7

The support is

$$S_{\mathbf{X}} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, 0 \leq x_2 \leq 3\}.$$



The density is normalized:

$$\int_0^\infty \int_0^3 \frac{1}{3} e^{-x_1} dx_2 dx_1 = \int_0^\infty e^{-x_1} dx_1 = 1.$$

For a region A , $P(\mathbf{X} \in A) = \iint_A f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$. The required region is $0 \leq x_2 \leq 3$ and $0 \leq x_1 \leq 4 - x_2$, so integrating with respect to x_1 first avoids splitting the region:

$$\begin{aligned} P(X_1 + X_2 \leq 4) &= \int_0^3 \int_0^{4-x_2} \frac{1}{3} e^{-x_1} dx_1 dx_2 \\ &= \frac{1}{3} \int_0^3 \left(1 - e^{-(4-x_2)}\right) dx_2 \\ &= \frac{1}{3} [x_2 - e^{-4} e^{x_2}]_0^3 \\ &= \boxed{1 - \frac{1}{3} e^{-1} + \frac{1}{3} e^{-4} \approx 0.8835.} \end{aligned}$$

Equivalently, reversing the integration order requires two pieces:

$$\int_0^1 \int_0^3 \frac{1}{3} e^{-x_1} dx_2 dx_1 + \int_1^4 \int_0^{4-x_1} \frac{1}{3} e^{-x_1} dx_2 dx_1.$$