Topics covered in this tutorial

- Conditional PMFs/PDFs and expectations
- Gaussian conditioning and independence
- Gaussian vectors, covariance and correlation matrices
- Affine transformations and cross-covariance

Complete Questions 1–5 before the tutorial test (normally around 16:00–16:15). Questions 6–8 are retained at the end for further practice.

Onderwerpe wat in hierdie tutoriaal gedek word

- Voorwaardelike WMF's/WDF's en verwagtings
- Gaussiese kondisionering en onafhanklikheid
- Gaussiese vektore, kovariansie- en korrelasiematrikse
- Affiene transformasies en kruiskovariansie

Voltooi Vrae 1–5 voor die tutoriaaltoets (gewoonlik omstreeks 16:00–16:15). Vrae 6–8 word aan die einde vir verdere oefening behou.

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1. Let $\mathbf{X} = [X_1\ X_2]'$ be a discrete random vector. Each component takes values in $\{1, 2, 3\}$, so its support is

Laat $\mathbf{X} = [X_1\ X_2]'$ 'n diskrete toevalsvektor wees. Elke komponent neem waardes in $\{1, 2, 3\}$ aan, dus is die steunversameling

$$S_{\mathbf{X}} = \{\mathbf{x} = [x_1\ x_2]' : x_1, x_2 \in \{1, 2, 3\}\}.$$

The joint probability mass function is

Die gesamentlike waarskynlikheidsmassafunksie is

$$P_{\mathbf{X}}(\mathbf{x}) = P_{\mathbf{X}}([x_1\ x_2]') = P_{X_1,X_2}(x_1, x_2) = \begin{cases} \dfrac{x_1 + x_2}{36}, & \mathbf{x} \in S_{\mathbf{X}}, \\ 0, & \mathbf{x} \notin S_{\mathbf{X}}. \end{cases}$$

(a) Find the complete PMF of $S = X_1 + X_2$. Hence find $P(S = 3)$.

(b) Find the marginal PMF $P_{X_2}(j)$ and the conditional PMF $P_{X_1|X_2}(i \mid j)$ for $j \in \{1, 2, 3\}$.

(c) Calculate $\mathrm{E}[X_1 \mid X_2 = j]$. Use this conditional expected value to argue whether X_1 and X_2 are independent.

(d) Use the PMF to calculate $\boldsymbol{\mu}_{\mathbf{X}}$, $\mathbf{R}_{\mathbf{X}} = \mathrm{E}[\mathbf{X}\mathbf{X}']$ and $\mathbf{C}_{\mathbf{X}}$. Use symmetry to reduce the calculations.

(a) Vind die volledige WMF van $S = X_1 + X_2$. Bepaal dus $P(S = 3)$.

(b) Vind die marginale WMF $P_{X_2}(j)$ en die voorwaardelike WMF $P_{X_1|X_2}(i \mid j)$ vir $j \in \{1, 2, 3\}$.

(c) Bereken $\mathrm{E}[X_1 \mid X_2 = j]$. Gebruik hierdie voorwaardelike verwagtingswaarde om te motiveer of X_1 en X_2 onafhanklik is.

(d) Gebruik die WMF om $\boldsymbol{\mu}_{\mathbf{X}}$, $\mathbf{R}_{\mathbf{X}} = \mathrm{E}[\mathbf{X}\mathbf{X}']$ en $\mathbf{C}_{\mathbf{X}}$ te bereken. Gebruik simmetrie om die berekeninge te verminder.

2. The continuous random variables X and Y have joint PDF

Die kontinue toevalsveranderlikes X en Y het gesamentlike WDF

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2}, & -1 \le x \le y \le 1, \\ 0, & \text{otherwise / andersins.} \end{cases}$$

(a) Find the marginal PDF $f_Y(y)$.

(b) Find $f_{X|Y}(x \mid y)$ and state the values of y for which the conditional-density formula is valid.

(c) Calculate $\mathrm{E}[X \mid Y = y]$ and $\mathrm{E}[Y - X \mid Y = y]$.

(d) Use iterated expectation to calculate $\mathrm{E}[X]$.

(a) Vind die marginale WDF $f_Y(y)$.

(b) Vind $f_{X|Y}(x \mid y)$ en meld die waardes van y waarvoor die voorwaardelike-digtheidsformule geldig is.

(c) Bereken $\mathrm{E}[X \mid Y = y]$ en $\mathrm{E}[Y - X \mid Y = y]$.

(d) Gebruik herhaalde verwagting om $\mathrm{E}[X]$ te bereken.

3. A temperature-monitoring system records temperature T (in °C) and a sensor voltage W (in V). They are jointly Gaussian, with the means and *standard deviations* below. An alarm occurs on the event $I = \{T > 38,\ W > 10\}$.

'n Temperatuurmoniteringstelsel meet temperatuur T (in °C) en 'n sensorspanning W (in V). Hulle is gesamentlik Gaussies, met die gemiddeldes en *standaardafwykings* hieronder. 'n Alarm vind plaas tydens die gebeurtenis $I = \{T > 38,\ W > 10\}$.

$$\mu_T = 37, \quad \sigma_T = 1, \qquad \mu_W = 7, \quad \sigma_W = 2.$$

(a) First assume $\rho_{T,W} = 0$. Explain why the individual probabilities can be multiplied as follows, then calculate $P(I)$.

(a) Aanvaar eers $\rho_{T,W} = 0$. Verduidelik waarom die afsonderlike waarskynlikhede soos volg vermenigvuldig kan word, en bereken dan $P(I)$.

$$P(I) = P(T > 38,\ W > 10) = P(T > 38)\,P(W > 10).$$

(b) Now use $\rho_{T,W} = 1/\sqrt{2}$ for the remaining parts. Find the conditional mean, variance and PDF of W given $T = t$.

(c) Express $P(I \mid T = t)$ in terms of Φ (the standard Gaussian CDF) for *all* real t. No numerical evaluation is required.

(d) Compare the conditional variance with $\mathrm{Var}[W]$. What does this say about the uncertainty in W once T is known?

(b) Gebruik nou $\rho_{T,W} = 1/\sqrt{2}$ vir die oorblywende dele. Vind die voorwaardelike gemiddelde, variansie en WDF van W gegee $T = t$.

(c) Druk $P(I \mid T = t)$ in terme van Φ (die standaard-Gaussiese KVF) uit vir *alle* reële t. Geen numeriese berekening word vereis nie.

(d) Vergelyk die voorwaardelike variansie met $\mathrm{Var}[W]$. Wat sê dit oor die onsekerheid in W wanneer T bekend is?

4. $\mathbf{X}$ is a 3-dimensional Gaussian random vector with mean vector $\boldsymbol{\mu}_\mathbf{X} = [4\ 8\ 6]'$ and covariance matrix

$\mathbf{X}$ is 'n 3-dimensionele Gaussiese toevalsvektor met gemiddeldevektor $\boldsymbol{\mu}_\mathbf{X} = [4\ 8\ 6]'$ en kovariansiematriks

$$\mathbf{C}_\mathbf{X} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}.$$

(a) Calculate the correlation matrix $\mathbf{R}_\mathbf{X} = \mathrm{E}[\mathbf{X}\mathbf{X}']$.

(b) Find the joint PDF $f_{X_1,X_2}(x_1, x_2)$.

(c) Calculate $P(X_1 > 8)$. Use $\Phi(2) = 0.9772$.

(d) Are X_1, X_2, X_3 independent? Justify using the covariance matrix and the Gaussian assumption.

(a) Bereken die korrelasiematriks $\mathbf{R}_\mathbf{X} = \mathrm{E}[\mathbf{X}\mathbf{X}']$.

(b) Vind die gesamentlike WDF $f_{X_1,X_2}(x_1, x_2)$.

(c) Bereken $P(X_1 > 8)$. Gebruik $\Phi(2) = 0.9772$.

(d) Is X_1, X_2, X_3 onafhanklik? Motiveer met die kovariansiematriks en die Gaussiese aanname.

5. Given the Gaussian random vector $\mathbf{X}$ in Question 4, define $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where

Gegewe die Gaussiese toevalsvektor $\mathbf{X}$ in Vraag 4, definieer $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, waar

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$$

Calculate:

(a) $\boldsymbol{\mu}_\mathbf{Y}$ **(b)** $\mathbf{C}_\mathbf{Y}$ **(c)** $\mathbf{R}_\mathbf{Y}$

(d) $P(-1 \le Y_2 \le 1)$ **(e)** $\mathbf{C}_{\mathbf{XY}}$ **(f)** $\mathbf{R}_{\mathbf{XY}}$

(g) ρ_{X_1,Y_2}

For (d), briefly justify why Y_2 is Gaussian. An answer in terms of Φ is sufficient.

(h) Can $\mathbf{C}_\mathbf{X}$ be recovered here using $\mathbf{A}^{-1}\mathbf{C}_\mathbf{Y}(\mathbf{A}^{-1})'$? Explain using the dimensions of $\mathbf{A}$.

Bereken:

(a) $\boldsymbol{\mu}_\mathbf{Y}$ **(b)** $\mathbf{C}_\mathbf{Y}$ **(c)** $\mathbf{R}_\mathbf{Y}$

(d) $P(-1 \le Y_2 \le 1)$ **(e)** $\mathbf{C}_{\mathbf{XY}}$ **(f)** $\mathbf{R}_{\mathbf{XY}}$

(g) ρ_{X_1,Y_2}

Motiveer vir (d) kortliks waarom Y_2 Gaussies is. 'n Antwoord in terme van Φ is voldoende.

(h) Kan $\mathbf{C}_\mathbf{X}$ hier herwin word met $\mathbf{A}^{-1}\mathbf{C}_\mathbf{Y}(\mathbf{A}^{-1})'$? Verduidelik aan die hand van $\mathbf{A}$ se afmetings.

Further practice

These questions are retained as further practice alongside the main tutorial (Questions 1–5).

Verdere oefening

Hierdie vrae word as verdere oefening saam met die hooftutoriaal (Vrae 1–5) behou.

6. The random vector $\mathbf{X} = [X_1\ X_2\ X_3]'$ has joint PDF

Die toevalsvektor $\mathbf{X} = [X_1\ X_2\ X_3]'$ het gesamentlike WDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} (x_1 + 2x_2 + 3x_3)/3, & 0 \le x_1, x_2, x_3 \le 1, \\ 0, & \text{otherwise / andersins.} \end{cases}$$

(a) Find $f_{X_1,X_2}(x_1, x_2)$ and $f_{X_3}(x_3)$. Is the subvector $[X_1\ X_2]'$ independent of X_3? Use factorisation to justify your answer.

(b) Derive $F_{\mathbf{X}}(\mathbf{x})$ for $0 \le x_1, x_2, x_3 \le 1$. Calculate $F_{\mathbf{X}}(0.3, 0.5, 1)$.

(c) Calculate $\mathrm{E}[X_1X_2X_3]$ directly from the joint PDF.

(a) Vind $f_{X_1,X_2}(x_1, x_2)$ en $f_{X_3}(x_3)$. Is die subvektor $[X_1\ X_2]'$ onafhanklik van X_3? Gebruik faktorisering om jou antwoord te motiveer.

(b) Lei $F_{\mathbf{X}}(\mathbf{x})$ af vir $0 \le x_1, x_2, x_3 \le 1$. Bereken $F_{\mathbf{X}}(0.3, 0.5, 1)$.

(c) Bereken $\mathrm{E}[X_1X_2X_3]$ direk uit die gesamentlike WDF.

7. Three tasks start simultaneously. Their completion times T_1, T_2, T_3 are independent and identically distributed, each uniform on $[0, 2]$ seconds. Let $M = \max(T_1, T_2, T_3)$ and $L = \min(T_1, T_2, T_3)$.

(a) Derive the complete CDF and PDF of M, starting from the event that all three tasks finish by time t.

(b) Derive the complete CDF and PDF of L.

(c) Find the probability that all tasks finish within 1.5 s, and the probability that at least one finishes within 0.5 s.

(d) Use independence to find $\mathrm{E}[T_1T_2T_3]$.

Drie take begin gelyktydig. Hul voltooiingstye T_1, T_2, T_3 is onafhanklik en identies verdeel, elk uniform op $[0, 2]$ sekondes. Laat $M = \max(T_1, T_2, T_3)$ en $L = \min(T_1, T_2, T_3)$ wees.

(a) Lei die volledige KVF en WDF van M af, vanaf die gebeurtenis dat al drie take teen tyd t klaar is.

(b) Lei die volledige KVF en WDF van L af.

(c) Vind die waarskynlikheid dat alle take binne 1.5 s klaar is, en die waarskynlikheid dat minstens een binne 0.5 s klaar is.

(d) Gebruik onafhanklikheid om $\mathrm{E}[T_1T_2T_3]$ te vind.

8. X_1 and X_2 are independent, each uniform on $[0, 1]$. Define $Y_1 = X_1 + X_2 + 1$ and $Y_2 = X_1 - X_2$.

(a) Write $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$. Find $\mathbf{A}^{-1}$ and $\det(\mathbf{A})$, and express x_1, x_2 in terms of y_1, y_2.

(b) Sketch the support of $\mathbf{Y}$ and derive its joint PDF using the vector-transformation formula.

(c) Given $\mathbf{C_X} = \frac{1}{12}\mathbf{I}_2$, where $\mathbf{I}_2$ is the 2×2 identity matrix, calculate $\mathbf{C_Y}$. Recover $\mathbf{C_X}$ using $\mathbf{A}^{-1}$. Are Y_1 and Y_2 independent? Justify using the support in (b).

(d) Consider the *separate* transformation $V_i = 2X_i + 1$, $i = 1, 2$. Express $F_{\mathbf{V}}(v_1, v_2)$ in terms of $F_{\mathbf{X}}$, then give the joint PDF and support of $\mathbf{V}$.

X_1 en X_2 is onafhanklik, elk uniform op $[0, 1]$. Definieer $Y_1 = X_1 + X_2 + 1$ en $Y_2 = X_1 - X_2$.

(a) Skryf $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$. Vind $\mathbf{A}^{-1}$ en $\det(\mathbf{A})$, en druk x_1, x_2 in terme van y_1, y_2 uit.

(b) Skets die steunversameling van $\mathbf{Y}$ en lei sy gesamentlike WDF af met die vektortransformasieformule.

(c) Gegee $\mathbf{C_X} = \frac{1}{12}\mathbf{I}_2$, waar $\mathbf{I}_2$ die 2×2-identiteitsmatriks is, bereken $\mathbf{C_Y}$. Herwin $\mathbf{C_X}$ met $\mathbf{A}^{-1}$. Is Y_1 en Y_2 onafhanklik? Motiveer met die steunversameling in (b).

(d) Beskou die *afsonderlike* transformasie $V_i = 2X_i + 1$, $i = 1, 2$. Druk $F_{\mathbf{V}}(v_1, v_2)$ in terme van $F_{\mathbf{X}}$ uit, en gee dan die gesamentlike WDF en steunversameling van $\mathbf{V}$.