

Question 1

(a) **Method 1: analytical evaluation.** Because U and V are independent, the PDF of their sum is the **convolution of their PDFs**: $f_Z(z) = (f_U * f_V)(z)$. The integral below and the flip-and-slide method on the next page evaluate the same convolution. The individual densities are

$$f_U(u) = \begin{cases} 1/3, & 2 \leq u \leq 5, \\ 0, & \text{otherwise,} \end{cases} \quad f_V(v) = \begin{cases} 1/2, & 0 \leq v \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

The general formula and the overlap conditions are

$$f_Z(z) = \int_{-\infty}^{\infty} f_U(u) f_V(z-u) du,$$

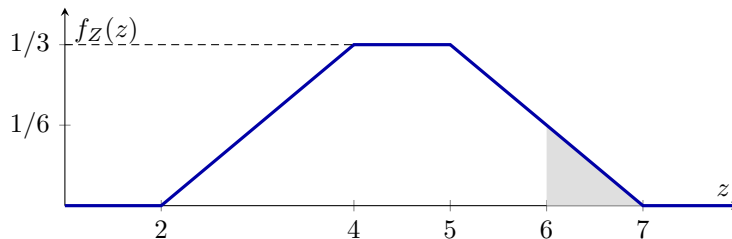
$$2 \leq u \leq 5, \quad 0 \leq z-u \leq 2 \quad \implies \quad \max(2, z-2) \leq u \leq \min(5, z).$$

There is overlap only for $2 \leq z \leq 7$. Within this support:

$$\begin{aligned} 2 \leq z \leq 4: \quad f_Z(z) &= \int_2^z \frac{1}{6} du = \frac{z-2}{6}, \\ 4 < z \leq 5: \quad f_Z(z) &= \int_{z-2}^z \frac{1}{6} du = \frac{1}{3}, \\ 5 < z \leq 7: \quad f_Z(z) &= \int_{z-2}^5 \frac{1}{6} du = \frac{7-z}{6}. \end{aligned}$$

Hence the complete density is

$$f_Z(z) = \begin{cases} (z-2)/6, & 2 \leq z \leq 4, \\ 1/3, & 4 < z \leq 5, \\ (7-z)/6, & 5 < z \leq 7, \\ 0, & \text{otherwise.} \end{cases}$$



(b) **Method 1: integration.** The required probability is the shaded tail area:

$$P(Z > 6) = \int_6^{\infty} f_Z(z) dz = \int_6^7 \frac{7-z}{6} dz = \frac{1}{6} \left[7z - \frac{z^2}{2} \right]_6^7 = \boxed{\frac{1}{12}}.$$

Question 1: alternative graphical solutions

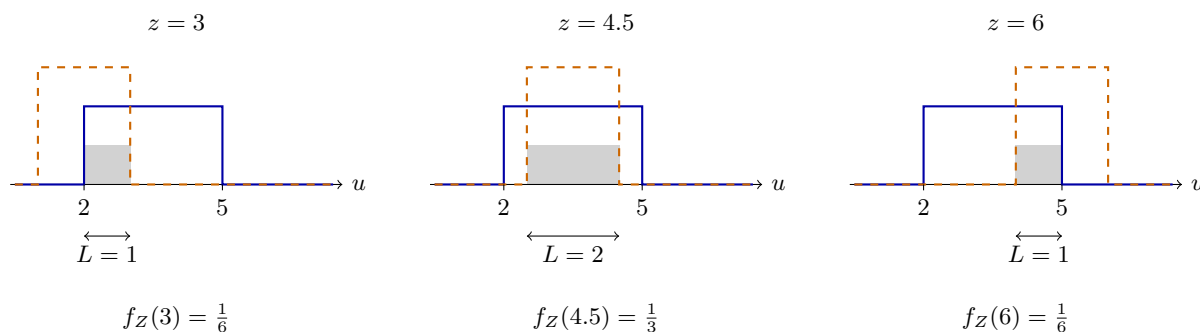
(a) **Method 2: convolution by flipping and sliding.** Hold $f_U(u)$ fixed: it is a rectangle of height $1/3$ on $[2, 5]$. Treat u as the horizontal plotting coordinate and z as the shift parameter:

$$\underbrace{f_V(u)}_{\text{support } [0,2]} \xrightarrow{\text{flip about } u=0} \underbrace{f_V(-u)}_{\text{support } [-2,0]} \xrightarrow{\text{shift right by } z} \underbrace{f_V(z-u)}_{\text{support } [z-2,z]} .$$

The moving rectangle still has height $1/2$. For each fixed z , multiply the two functions point by point and integrate the product over u :

$$(f_U * f_V)(z) = \int_{-\infty}^{\infty} f_U(u) f_V(z-u) du = \underbrace{\frac{1}{3} \cdot \frac{1}{2}}_{\text{product height } 1/6} \underbrace{L(z)}_{\text{horizontal overlap length}} .$$

Thus we need the *area under the product*, not the common area of the two original rectangles. The snapshots show the fixed PDF in solid blue, the flipped-and-shifted PDF in dashed orange, and their product shaded grey.



Sliding from left to right gives the same three nonzero pieces as Method 1:

Shift z	Position of moving support	$L(z)$	$f_Z(z) = L(z)/6$
$z < 2$	Not yet overlapping	0	0
$2 \leq z \leq 4$	Entering the fixed rectangle	$z - 2$	$(z - 2)/6$
$4 < z \leq 5$	Entire width 2 is inside	2	$1/3$
$5 < z \leq 7$	Leaving the fixed rectangle	$7 - z$	$(7 - z)/6$
$z > 7$	No longer overlapping	0	0

At $z = 2$ and $z = 7$ the supports only touch, so the overlap length is zero. The overlap grows linearly, stays constant, then shrinks linearly: this explains the trapezoidal PDF sketched on the preceding page.

(b) **Method 2: geometry.** The region under f_Z for $z > 6$ is a right triangle; the density is zero beyond $z = 7$. Its base is $7 - 6 = 1$ and its height is $f_Z(6) = (7 - 6)/6 = 1/6$. Therefore, without integration,

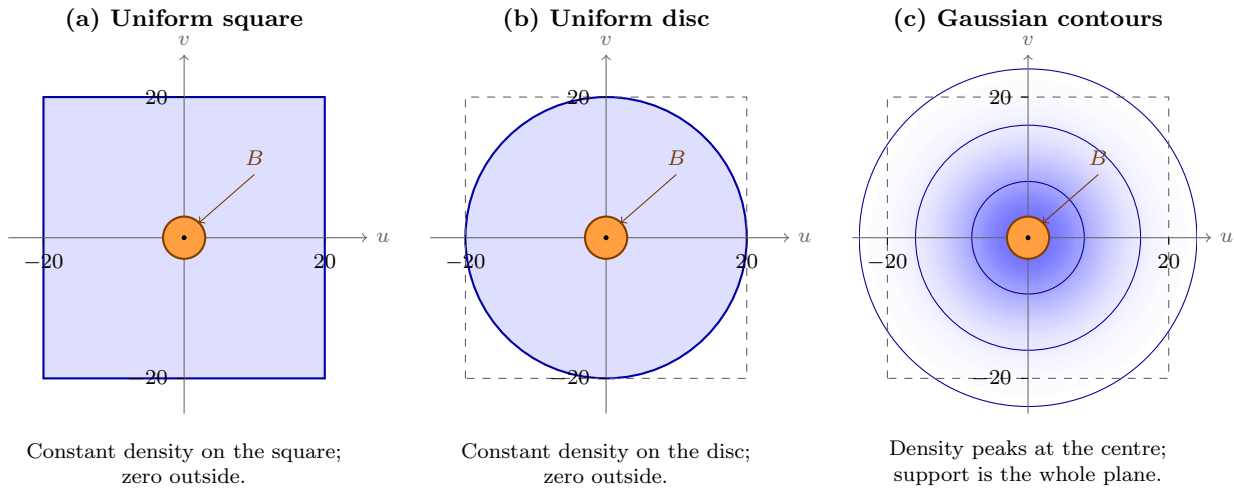
$$P(Z > 6) = \text{triangle area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 1 \times \frac{1}{6} = \boxed{\frac{1}{12}} .$$

This is the same shaded tail as in Method 1. Because Z is continuous, including or excluding the single point $z = 6$ does not change the probability.

Question 2

The Cartesian plane is labelled (u, v) here (the usual x - y plane), matching the random variables U, V in the question. The bullseye is the *filled disc* $B = \{U^2 + V^2 \leq 9\}$, not just its circular boundary. For any of the three models,

$$P(B) = \iint_{u^2+v^2 \leq 9} f_{U,V}(u, v) du dv.$$



All axes are in cm and use the same scale. Orange marks the radius-3 bullseye whose probability is calculated. The square board has side 40; dashed squares in (b)–(c) are only board references. In (c), darker blue means higher density; the circles at radii 8, 16, 24 are equal-density contours, *not* support boundaries or integration limits. The finite viewing window does not truncate the Gaussian model.

(a) Each marginal is uniform on $[-20, 20]$ with density $1/40$. By independence,

$$f_{U,V}(u, v) = \begin{cases} 1/1600, & -20 \leq u, v \leq 20, \\ 0, & \text{otherwise.} \end{cases}$$

The joint density is constant throughout the bullseye, so

$$P(B) = \frac{\text{bullseye area}}{\text{square area}} = \frac{\pi(3)^2}{40^2} = \boxed{\frac{9\pi}{1600} \approx 0.0177}.$$

(b) Uniformity over the disc gives

$$f_{U,V}(u, v) = \begin{cases} 1/(400\pi), & u^2 + v^2 \leq 400, \\ 0, & \text{otherwise,} \end{cases}$$

and therefore

$$P(B) = \frac{\text{bullseye area}}{\text{radius-20 disc area}} = \frac{\pi(3)^2}{\pi(20)^2} = \boxed{\frac{9}{400} = 0.0225}.$$

(c) The Gaussian model is defined over the whole plane, including possible misses of the board. This is an unconditional bullseye probability. For $\sigma = 8$, independence gives

$$f_{U,V}(u, v) = f_U(u)f_V(v) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{u^2 + v^2}{2\sigma^2}\right).$$

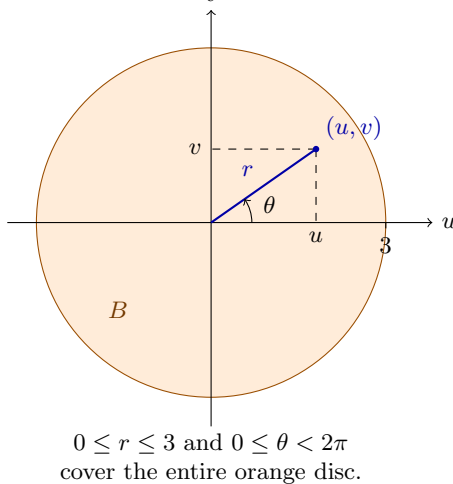
Unlike (a)–(b), an area ratio alone cannot give this probability: points near the centre carry more density than points farther away. We must integrate the density over the orange disc. Polar coordinates simplify the circular region and the expression $u^2 + v^2$, as shown next.

Question 2(c): why polar coordinates help

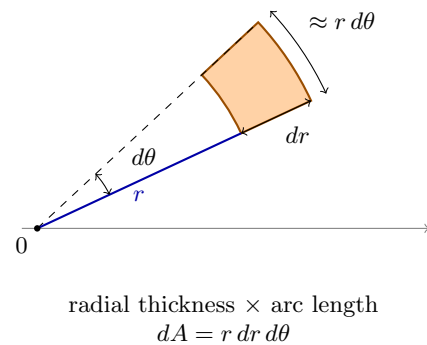
Instead of locating a point by its horizontal and vertical coordinates, use its distance $r \geq 0$ from the origin and angle θ measured anticlockwise from the positive u -axis. The right triangle gives

$$u = r \cos \theta, \quad v = r \sin \theta, \quad u^2 + v^2 = r^2.$$

The bullseye in polar coordinates



A small polar area patch (enlarged)



The limits: at a fixed angle, increasing r from 0 to 3 traces one radius of the bullseye. Letting θ run from 0 to 2π rotates that radius through a full turn, covering all four quadrants. Angles are in *radians*; negative u or v values are supplied by the signs of $\cos \theta$ and $\sin \theta$, not by taking $r < 0$.

The extra factor r : the enlarged patch has radial thickness dr and arc length approximately $r d\theta$. Multiplying these lengths gives $dA = r dr d\theta$ in the differential limit. The same angular step covers a longer arc farther from the origin, so replacing $du dv$ by just $dr d\theta$ would undercount area there.

Applying this to the Gaussian density:

$$\underbrace{\{(u, v) : u^2 + v^2 \leq 9\}}_{\text{disc in Cartesian coordinates}} \longleftrightarrow \underbrace{0 \leq r \leq 3, \quad 0 \leq \theta < 2\pi}_{\text{constant bounds in polar coordinates}}.$$

The density depends only on distance r , not on direction θ . Substitute $u^2 + v^2 = r^2$ and $du dv = r dr d\theta$:

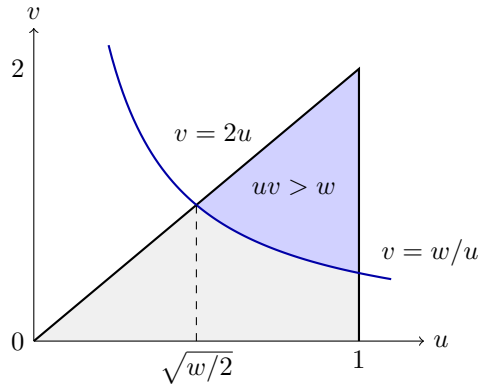
$$\begin{aligned} P(B) &= \int_0^{2\pi} \int_0^3 \frac{1}{2\pi\sigma^2} e^{-r^2/(2\sigma^2)} r dr d\theta \\ &= \frac{1}{\sigma^2} \int_0^3 r e^{-r^2/(2\sigma^2)} dr \\ &= \left[-e^{-r^2/(2\sigma^2)} \right]_0^3 = 1 - e^{-9/(2 \cdot 8^2)} = \boxed{1 - e^{-9/128} \approx 0.0679}. \end{aligned}$$

Equivalent ring intuition: a thin full ring of radius r and thickness dr has area approximately $2\pi r dr$. Its probability is the density at that radius times this area. Adding the rings from $r = 0$ to 3 gives the same integral:

$$P(B) = \int_0^3 \underbrace{\frac{1}{2\pi\sigma^2} e^{-r^2/(2\sigma^2)}}_{\text{density on the ring}} \underbrace{2\pi r dr}_{\text{ring area}}.$$

Question 3

The support is the triangle bounded by $v = 0$, $v = 2u$ and $u = 1$. Its area is 1, consistent with a constant density of 1. Since $W = UV$, its support is $S_W = [0, 2]$. For $0 < w < 2$, use the complement of the region $uv \leq w$. The hyperbola $v = w/u$ meets $v = 2u$ at $u = \sqrt{w/2}$.



The sketch illustrates $w = 1/2$; the following limits hold for every $0 < w < 2$.

$$\begin{aligned}
 F_W(w) &= P(W \leq w) = 1 - P(UV > w) \\
 &= 1 - \int_{\sqrt{w/2}}^1 \int_{w/u}^{2u} 1 \, dv \, du \\
 &= 1 - \int_{\sqrt{w/2}}^1 \left(2u - \frac{w}{u}\right) du \\
 &= 1 - [u^2 - w \ln u]_{\sqrt{w/2}}^1 = \frac{w}{2} \left(1 - \ln \frac{w}{2}\right).
 \end{aligned}$$

Thus

$$F_W(w) = \begin{cases} 0, & w \leq 0, \\ \frac{w}{2} \left(1 - \ln \frac{w}{2}\right), & 0 < w < 2, \\ 1, & w \geq 2. \end{cases} \quad f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} \frac{1}{2} \ln \frac{2}{w}, & 0 < w < 2, \\ 0, & \text{otherwise.} \end{cases}$$

The CDF is continuous at 0 because $w \ln w \rightarrow 0$ as $w \downarrow 0$. The density grows without bound near 0, but is integrable and has total area 1; there is no point mass at 0. Its assigned value at a single endpoint does not affect probabilities.

Question 4

Let X be the transmitted bit, W the independent Gaussian noise (mean 0.2, standard deviation 0.3), and $V = X + W$ the received amplitude. The decoded bit $\hat{X}$ follows the receiver's rule:

$$\hat{X} = \begin{cases} 0, & V < 0.6, \\ 1, & V \geq 0.6. \end{cases} \quad E = \{\hat{X} \neq X\}.$$

An error always means *decoded bit differs from transmitted bit*. However, the received amplitudes that cause an error depend on which bit was sent:

Sent bit	Wrong decoded bit	Error condition on V	Tail needed
$X = 1$	$\hat{X} = 0$	$V < 0.6$	Lower (left) tail
$X = 0$	$\hat{X} = 1$	$V \geq 0.6$	Upper (right) tail

These are two disjoint ways for the same error event to occur:

$$E = \{X = 0, V \geq 0.6\} \cup \{X = 1, V < 0.6\}.$$

Write Φ for the standard Gaussian CDF, which gives a *left-tail* probability. Upper-tail probabilities therefore require $1 - \Phi(\cdot)$.

(a) Error given that a 1 was sent. The receiver makes an error only if it decodes 0, meaning $V < 0.6$. Given $X = 1$, the received amplitude has mean $1 + 0.2 = 1.2$ and standard deviation 0.3. Thus

$$\begin{aligned} P(E | X = 1) &= P(\hat{X} = 0 | X = 1) = P(V < 0.6 | X = 1) \\ &= \Phi\left(\frac{\text{threshold} - \text{mean}}{\text{standard deviation}}\right) = \Phi\left(\frac{0.6 - 1.2}{0.3}\right) = \boxed{\Phi(-2) \approx 0.0228}. \end{aligned}$$

(b) Error for a randomly transmitted bit. We must account for both possible transmitted bits, not just the case in (a).

Step 1: identify the error when a 0 is sent. An error now means decoding 1, so the received amplitude must be *at or above* the threshold:

$$\begin{aligned} P(E | X = 0) &= P(\hat{X} = 1 | X = 0) = P(V \geq 0.6 | X = 0) \\ &= 1 - P(V < 0.6 | X = 0). \end{aligned}$$

Here $V < 0.6$ would be a *correct* decision when 0 was sent. Subtracting that left-tail probability from 1 leaves the right-tail error probability. This is why the $1 -$ is needed here, but not in (a).

Step 2: calculate this conditional error probability. Given $X = 0$, $V = W$ has mean 0.2 and standard deviation 0.3. Hence

$$\begin{aligned} P(E | X = 0) &= 1 - \Phi\left(\frac{\text{threshold} - \text{mean}}{\text{standard deviation}}\right) \\ &= 1 - \Phi\left(\frac{0.6 - 0.2}{0.3}\right) = 1 - \Phi(4/3) \approx 0.0912. \end{aligned}$$

The threshold itself has probability zero under a continuous Gaussian law, so strict versus non-strict inequalities do not change these values.

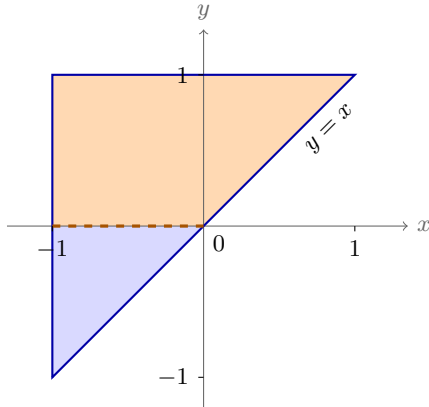
Step 3: average over which bit was sent. The transmitted-bit events $X = 0$ and $X = 1$ form a partition, each with probability $1/2$. By the law of total probability,

$$\begin{aligned} P(E) &= P(E | X = 0)P(X = 0) + P(E | X = 1)P(X = 1) \\ &= \frac{1}{2}[1 - \Phi(4/3)] + \frac{1}{2}\Phi(-2) \\ &\approx \frac{1}{2}(0.0912) + \frac{1}{2}(0.0228) \approx \boxed{0.0570}. \end{aligned}$$

Question 5

The support is $-1 \leq x \leq y \leq 1$, a triangle of area 2; density $1/2$ therefore integrates to 1. The left sketch is a top view of this constant joint PDF. The event $B = \{Y > 0\}$ is the upper half-plane, but only its intersection with the triangular support contributes probability (orange).

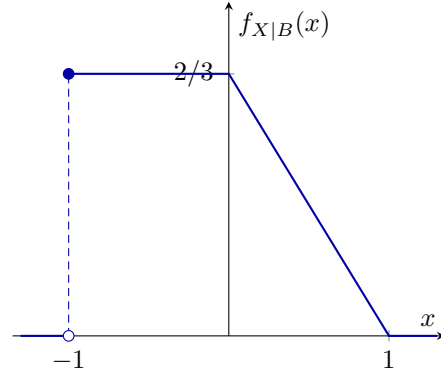
Joint PDF and the retained region



Orange: $S_{X,Y} \cap B$.

Blue: the part excluded by conditioning.

Resulting conditional PDF



The density after restricting to B and normalising the retained probability.

The density is $1/2$ everywhere inside the whole triangle, including the orange part; the colour identifies the event, not a different density height. The dashed segment at $y = 0$ is excluded by $Y > 0$; a boundary line has zero probability here.

(a) Integrating the joint density over x gives

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \int_{-1}^y \frac{1}{2} dx = \frac{y+1}{2}, \quad -1 \leq y \leq 1.$$

Thus $f_Y(y) = (y+1)/2$ on $[-1, 1]$, and 0 otherwise.

(b) **First marginalise over the region $B = \{Y > 0\}$.** Integrate out Y , retaining only outcomes in the orange region. For $-1 \leq x \leq 0$, the allowed y interval is $(0, 1]$; for $0 < x \leq 1$, it is $[x, 1]$. Thus

$$\begin{aligned} -1 \leq x \leq 0: \quad & \int_0^1 f_{X,Y}(x,y) dy = \int_0^1 \frac{1}{2} dy = \frac{1}{2}, \\ 0 < x \leq 1: \quad & \int_x^1 f_{X,Y}(x,y) dy = \int_x^1 \frac{1}{2} dy = \frac{1-x}{2}. \end{aligned}$$

The contribution is 0 outside $[-1, 1]$. These are the *marginal contributions from outcomes in B , before normalisation*, not the ordinary marginal PDF $f_X(x)$, which includes all outcomes.

Next calculate $P(B)$. From part (a),

$$P(B) = \int_0^1 f_Y(y) dy = \int_0^1 \frac{y+1}{2} dy = \frac{3}{4}.$$

Finally divide the marginal contribution from B by $P(B)$.

$$f_{X|B}(x) = \frac{\int_0^1 f_{X,Y}(x,y) dy}{P(B)}, \quad -1 \leq x \leq 1.$$

Dividing the two contributions above by $3/4$ gives

$$f_{X|B}(x) = \begin{cases} \frac{2}{3}, & -1 \leq x \leq 0, \\ \frac{2(1-x)}{3}, & 0 < x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Question 6

The parameter restriction $b > 0$ is needed for the given CDF to define exponential distributions. Differentiate the joint CDF in the interior of its support:

$$\begin{aligned} f_{U,V}(u, v) &= \frac{\partial^2 F_{U,V}(u, v)}{\partial v \partial u} \\ &= \frac{\partial}{\partial v} [be^{-bu} - be^{-b(u+v)}] = b^2 e^{-b(u+v)}, \quad u, v \geq 0. \end{aligned}$$

The joint density is 0 otherwise.

First marginalise over the region $B = \{V > 1\}$. Integrate out V , retaining only outcomes in the orange region. This is the density contribution corresponding to an *intersection*. In this solution, define $f_{U \cap B}(u)$ as shorthand for this *unnormalised contribution*, not the ordinary marginal PDF:

$$\begin{aligned} f_{U \cap B}(u) &:= \int_1^\infty f_{U,V}(u, v) dv = \frac{d}{du} P(\{U \leq u\} \cap B) \\ &= \int_1^\infty b^2 e^{-b(u+v)} dv = be^{-bu} e^{-b}, \quad u > 0. \end{aligned}$$

Here $\{U \leq u\} \cap B$ means $U \leq u$ and $V > 1$ together. The subscript $U \cap B$ is a locally defined shorthand, not standard PDF notation. Its integral over u is $P(B)$, not 1. It is 0 for $u < 0$; the value at the single endpoint $u = 0$ does not affect probabilities.

Next calculate $P(B)$. Integrate the joint PDF directly over the orange region, $u \geq 0, v > 1$:

$$\begin{aligned} P(B) &= \int_0^\infty \int_1^\infty f_{U,V}(u, v) dv du = \int_0^\infty \int_1^\infty b^2 e^{-b(u+v)} dv du \\ &= [-e^{-bu}]_0^\infty [-e^{-bv}]_1^\infty = 1 \cdot e^{-b} = e^{-b}. \end{aligned}$$

Finally divide the marginal contribution from B by $P(B)$.

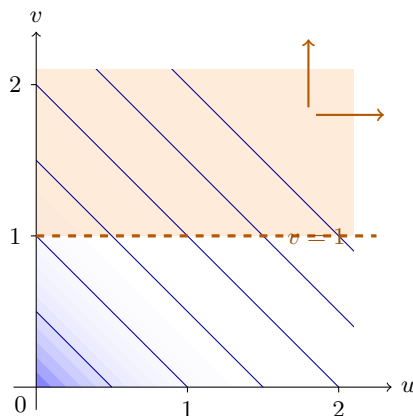
$$f_{U|B}(u) = \frac{f_{U \cap B}(u)}{P(B)} = \frac{be^{-bu} e^{-b}}{e^{-b}} = be^{-bu}, \quad u \geq 0.$$

$$f_{U|B}(u) = \begin{cases} be^{-bu}, & u \geq 0, \\ 0, & u < 0. \end{cases}$$

The joint PDF factorises as $(be^{-bu})(be^{-bv})$, so U and V are independent. Hence this conditional PDF also equals the ordinary marginal $f_U(u)$: knowing $V > 1$ does not change the distribution of U .

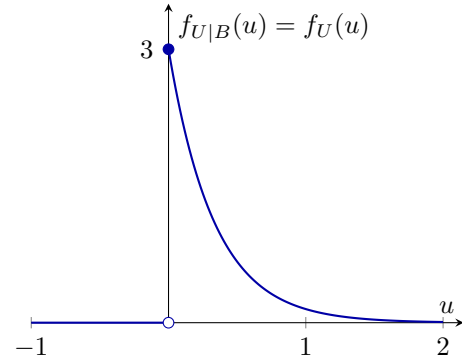
Sketches for $b = 3$. Orange marks the retained region $S_{U,V} \cap B = \{u \geq 0, v > 1\}$.

Joint PDF and the retained region



Orange: $S_{U,V} \cap B$, continuing up/right.
Contours are lines of constant $u + v$.

Conditional and original marginal PDFs



The two curves coincide: $3e^{-3u}$ for $u \geq 0$,
and zero for $u < 0$.

Darker blue indicates higher joint density $9e^{-3(u+v)}$; orange identifies the retained region, not a density level. The finite window does not bound the support. The dashed line $v = 1$ is excluded and has probability zero. For $b = 3$, the unbounded orange region has probability $e^{-3} \approx 0.0498$; its probability is not an area ratio.