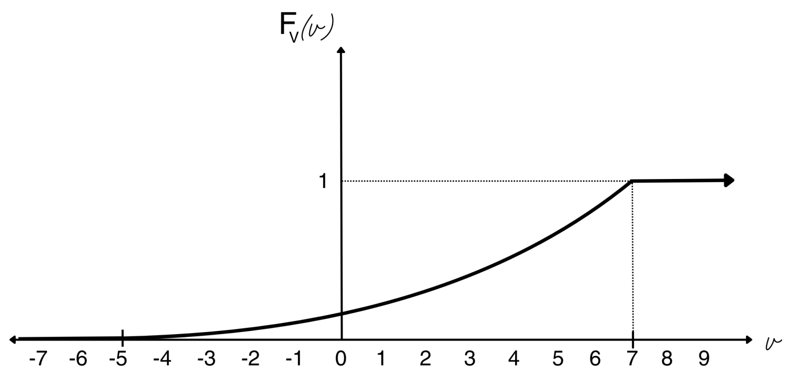
Question 1

For the CDF of continuous random variable V:

$$F_V(v) = \begin{cases} 0 & v < -5 \\ c(v+5)^2 & -5 \le v < 7 \\ 1 & v \ge 7 \end{cases}$$



(a) **Finding the constant c:**
For a valid CDF, we need continuity at the boundary points and $F_V(7^-) = 1$.
At $v = 7$:

$$\begin{aligned} F_V(7^-) = c(7+5)^2 &= 1 \\ c \cdot 12^2 &= 1 \\ 144c &= 1 \\ c &= \frac{1}{144} \end{aligned}$$

Verification: We can verify this is correct by checking that $F_V(-5) = 0$: $F_V(-5) = c(-5+5)^2 = c \cdot 0 = 0$

(b) **Finding $P(V > 4)$:**
Since 4 is in the range $[-5, 7)$, we use the middle piece of the CDF:

$$\begin{aligned} P(V > 4) &= 1 - P(V \le 4) \\ &= 1 - F_V(4) \\ &= 1 - c(4+5)^2 \\ &= \frac{7}{16} \\ &= 0.4375 \end{aligned}$$

(c) **Finding $P(-3 < V \le 0)$:**
Both -3 and 0 are in the range $[-5, 7)$:

$$\begin{aligned} P(-3 < V \le 0) &= F_V(0) - F_V(-3) \\ &= c(0+5)^2 - c(-3+5)^2 \\ &= \frac{7}{48} \\ &= 0.1458 \end{aligned}$$

(d) **Finding $P(|V| \leq 4)$:**
$P(|V| \leq 4) = P(-4 \leq V \leq 4)$
Since both -4 and 4 are in $[-5, 7)$:

$$\begin{aligned}
P(-4 \leq V \leq 4) &= F_V(4) - F_V(-4^-) \\
&= F_V(4) - F_V(-4) \\
&= c(4+5)^2 - c(-4+5)^2 \\
&= \frac{1}{144}(81) - \frac{1}{144}(1) \\
&= \frac{80}{144} \\
&= \frac{5}{9}
\end{aligned}$$

(e) **Finding a such that $P(V > a) = \frac{2}{3}$:**
We need $P(V > a) = 1 - F_V(a) = \frac{2}{3}$, so $F_V(a) = \frac{1}{3}$.
Since a must be in $[-5, 7)$ for a non-trivial solution:

$$\begin{aligned}
F_V(a) = c(a+5)^2 &= \frac{1}{3} \\
\frac{1}{144}(a+5)^2 &= \frac{1}{3} \\
(a+5)^2 &= 48 \\
a + 5 &= \pm\sqrt{48} = \pm 4\sqrt{3} \\
a &= -5 \pm 4\sqrt{3}
\end{aligned}$$

Since a must be in $[-5, 7)$, we take $a = -5 + 4\sqrt{3} \approx 1.93$.

Question 2

(a)

$$\begin{aligned}
P(X \leq 5) &= F_X(5) \\
&= F\left(\frac{5-8}{\sqrt{4}}\right) \\
&= \Phi(-1.5) \\
&= 1 - \Phi(1.5) \\
&= 0.0668
\end{aligned}$$

(b)

$$\begin{aligned}
P(4 < X \leq 6) &= F_X(6) - F_X(4) \\
&= \Phi\left(\frac{6-8}{\sqrt{4}}\right) - \Phi\left(\frac{4-8}{\sqrt{4}}\right) \\
&= \Phi(-1) - \Phi(-2) \\
&= 1 - \Phi(1) - (1 - \Phi(2)) \\
&= 0.1587 - 0.02275 \\
&= 0.13595
\end{aligned}$$

(c)

$$
\begin{aligned}
P(X > 9) &= 1 - F_X(9) \\
&= 1 - \Phi\left(\frac{9-8}{\sqrt{4}}\right) \\
&= 1 - \Phi(0.5) \\
&= 0.3085
\end{aligned}
$$

Question 3

Step 1: Find the distribution parameters
Given: Wind speed W is Gaussian with $\sigma_W^2 = 64$, so $\sigma_W = 8$. Also given: $P(W > 20) = 0.5$.
Since $P(W > 20) = 0.5$, this means that 20 is the median of the distribution. For a Gaussian distribution, the mean equals the median, so $\mu_W = 20$.
Step 2: Find $P(W > 30)$
Now we can calculate:

$$
\begin{aligned}
P(W > 30) &= 1 - P(W \leq 30) \\
&= 1 - \Phi\left(\frac{30-20}{8}\right) \\
&= 1 - \Phi(1.25) \\
&= 1 - 0.8944 \\
&= 0.1056
\end{aligned}
$$

Question 4

For the exponential distribution with CDF $F_X(x) = 1 - e^{-3x}$ for $x \geq 0$:

(a) **Mean number of battery failures per hour:**

The CDF indicates this is an exponential distribution with rate parameter $\lambda = 3$.

The corresponding PDF is: $f_X(x) = 3e^{-3x}$ for $x \geq 0$.

For an exponential distribution:

- Mean time between events $= \frac{1}{\lambda} = \frac{1}{3}$ hours
- Rate of events $= \lambda = 3$ events per hour

Therefore, the mean number of times per hour that the battery dies is **3 times**.

(b) **Probability of lasting through 15-minute meeting:**

15 minutes $= \frac{15}{60} = \frac{1}{4} = 0.25$ hours.

We need $P(X > 0.25)$:

$$
\begin{aligned}
P(X > 0.25) &= 1 - P(X \leq 0.25) \\
&= 1 - F_X(0.25) \\
&= 1 - (1 - e^{-3\times 0.25}) \\
&= e^{-0.75} \\
&= 0.4724
\end{aligned}
$$

So there's approximately a 47.2% chance the phone will last through the meeting.

Question 5

For the new triangular distribution of random variable Z with PDF shown in the figure:
The PDF can be written as:

$$f_Z(z) = \begin{cases} \frac{z}{9} & \text{for } 0 \leq z \leq 3 \\ \frac{6-z}{9} & \text{for } 3 < z \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Given: $m_1 = E[Z] = 3$ (which makes sense due to symmetry)
Finding the variance σ_Z^2:
We use the formula: $\sigma_Z^2 = E[Z^2] - (E[Z])^2 = m_2 - m_1^2$
Step 1: Calculate $m_2 = E[Z^2]$

$$\begin{aligned} m_2 = E[Z^2] &= \int_0^6 z^2 f_Z(z)\, dz \\ &= \int_0^3 z^2 \cdot \frac{z}{9}\, dz + \int_3^6 z^2 \cdot \frac{6-z}{9}\, dz \\ &= \frac{1}{9}\int_0^3 z^3\, dz + \frac{1}{9}\int_3^6 z^2(6-z)\, dz \\ &= \frac{1}{9}\int_0^3 z^3\, dz + \frac{1}{9}\int_3^6 (6z^2 - z^3)\, dz \end{aligned}$$

Evaluating the first integral:

$$\begin{aligned} \frac{1}{9}\int_0^3 z^3\, dz &= \frac{1}{9}\left[\frac{z^4}{4}\right]_0^3 \\ &= \frac{1}{9} \cdot \frac{81}{4} = \frac{9}{4} \end{aligned}$$

Evaluating the second integral:

$$\begin{aligned} \frac{1}{9}\int_3^6 (6z^2 - z^3)\, dz &= \frac{1}{9}\left[2z^3 - \frac{z^4}{4}\right]_3^6 \\ &= \frac{1}{9}\left[(2 \cdot 216 - \frac{1296}{4}) - (2 \cdot 27 - \frac{81}{4})\right] \\ &= \frac{1}{9}\left[432 - 324 - 54 + \frac{81}{4}\right] \\ &= \frac{1}{9}\left[54 + \frac{81}{4}\right] \\ &= \frac{1}{9} \cdot \frac{216 + 81}{4} \\ &= \frac{297}{36} = \frac{33}{4} \end{aligned}$$

Therefore:

$$m_2 = \frac{9}{4} + \frac{33}{4} = \frac{42}{4} = \frac{21}{2}$$

Step 2: Calculate the variance

$$\begin{aligned}
\sigma_Z^2 &= m_2 - m_1^2 \\
&= \frac{21}{2} - 3^2 \\
&= \frac{21}{2} - 9 \\
&= \frac{21-18}{2} \\
&= \frac{3}{2} = 1.5
\end{aligned}$$

Question 6

For the trapezoidal distribution representing delivery time (in hours):
The PDF can be written as:

$$f_Z(z) = \begin{cases} \frac{b}{2}z & \text{for } 0 \leq z \leq 2 \\ b & \text{for } 2 < z \leq 6 \\ b\left(\frac{8-z}{2}\right) = \frac{b(8-z)}{2} & \text{for } 6 < z \leq 8 \\ 0 & \text{otherwise} \end{cases}$$

(a) **Finding the constant b:**

The total area under the PDF must equal 1. We can calculate this as the area of a trapezoid:

Area $= \frac{1}{2}$(sum of parallel sides) $\times$ height

The parallel sides are the top and bottom of the trapezoid:

- Bottom base $= 8$ (width from 0 to 8)
- Top base $= 4$ (width from 2 to 6 where height is constant at b)
- Height $= b$

Alternatively, we can integrate:

$$\begin{aligned}
\int_0^8 f_Z(z)\,dz &= \int_0^2 \frac{bz}{2}\,dz + \int_2^6 b\,dz + \int_6^8 \frac{b(8-z)}{2}\,dz \\
&= \frac{b}{2}\left[\frac{z^2}{2}\right]_0^2 + b[z]_2^6 + \frac{b}{2}\left[8z - \frac{z^2}{2}\right]_6^8 \\
&= \frac{b}{2}\cdot 2 + b\cdot 4 + \frac{b}{2}[(64-32)-(48-18)] \\
&= b + 4b + \frac{b}{2}[32-30] \\
&= b + 4b + b \\
&= 6b = 1
\end{aligned}$$

Therefore: $b = \frac{1}{6}$

Using area (geometry): Decompose the shape into two right triangles and a rectangle.

$$\begin{aligned}
1 \;=\; \text{Area} &= \underbrace{\tfrac{1}{2}\cdot 2\cdot b}_{\text{left triangle } (0\to 2)} \;+\; \underbrace{4\cdot b}_{\text{rectangle } (2\to 6)} \;+\; \underbrace{\tfrac{1}{2}\cdot 2\cdot b}_{\text{right triangle } (6\to 8)} \\
&= b + 4b + b \;=\; 6b.
\end{aligned}$$

Thus, $6b = 1 \Rightarrow b = \frac{1}{6}$.

(b) **Probability of delivery within 5 hours:**

We need $P(Z \leq 5) = F_Z(5)$.

Since 5 is in the interval $(2, 6]$ where $f_Z(z) = b = \frac{1}{6}$:

$$\begin{aligned} P(Z \leq 5) &= \int_0^5 f_Z(z)\,dz \\ &= \int_0^2 \frac{z}{12}\,dz + \int_2^5 \frac{1}{6}\,dz \\ &= \frac{1}{12}\left[\frac{z^2}{2}\right]_0^2 + \frac{1}{6}[z]_2^5 \\ &= \frac{1}{12}\cdot 2 + \frac{1}{6}\cdot 3 \\ &= \frac{1}{6} + \frac{1}{2} \\ &= \frac{1+3}{6} = \frac{4}{6} = \frac{2}{3} \end{aligned}$$

Therefore, there is a $\frac{2}{3}$ or approximately 66.7% probability that the delivery will be completed within 5 hours.

Using area (geometry): The probability corresponds to the total shaded area under the PDF from 0 to 5. This region can be decomposed into:

$$\begin{aligned} P(Z \leq 5) &= \underbrace{\tfrac{1}{2}\cdot 2\cdot \tfrac{1}{6}}_{\text{left triangle } (0\to 2)} + \underbrace{3\cdot \tfrac{1}{6}}_{\text{rectangle } (2\to 5)} \\ &= \tfrac{1}{6} + \tfrac{3}{6} \\ &= \tfrac{4}{6} = \tfrac{2}{3}. \end{aligned}$$

Thus, the probability is again $\frac{2}{3}$ (66.7%).

(c) **Sketching the CDF $F_Z(z)$:**

To find the CDF, we integrate the PDF over each region:

For $0 \leq z \leq 2$:

$$F_Z(z) = \int_0^z \frac{t}{12}\,dt = \frac{1}{12}\cdot\frac{z^2}{2} = \frac{z^2}{24}$$

For $2 < z \leq 6$:

$$\begin{aligned} F_Z(z) &= \int_0^2 \frac{t}{12}\,dt + \int_2^z \frac{1}{6}\,dt \\ &= \frac{4}{24} + \frac{1}{6}(z-2) \\ &= \frac{1}{6} + \frac{z-2}{6} \\ &= \frac{z-1}{6} \end{aligned}$$

For $6 < z \leq 8$:

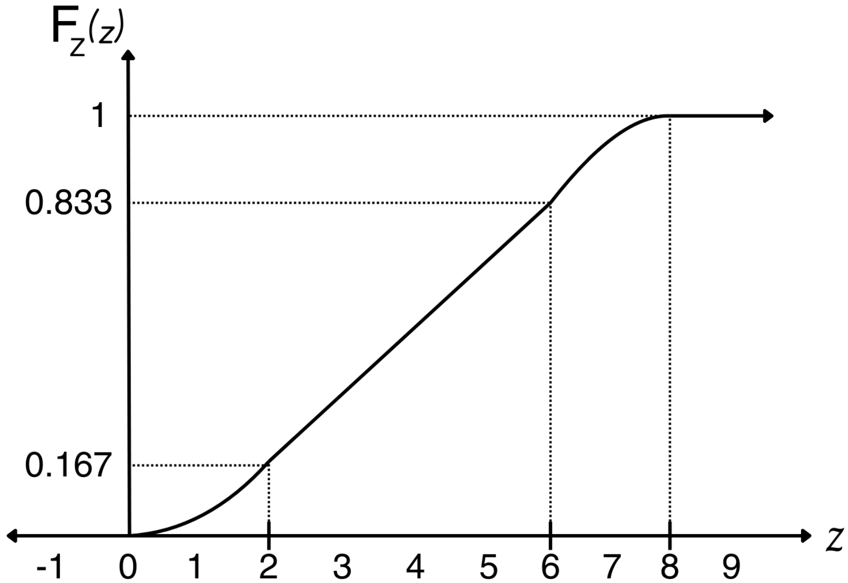
$$\begin{aligned}
F_Z(z) &= \int_0^2 \frac{t}{12}\,dt + \int_2^6 \frac{1}{6}\,dt + \int_6^z \frac{8-t}{12}\,dt \\
&= \frac{1}{6} + \frac{4}{6} + \frac{1}{12}\int_6^z (8-t)\,dt \\
&= \frac{5}{6} + \frac{1}{12}\left[8t - \frac{t^2}{2}\right]_6^z \\
&= \frac{5}{6} + \frac{1}{12}\left[(8z - \frac{z^2}{2}) - (48-18)\right] \\
&= \frac{5}{6} + \frac{1}{12}\left[8z - \frac{z^2}{2} - 30\right] \\
&= \frac{5}{6} + \frac{8z - \frac{z^2}{2} - 30}{12} \\
&= \frac{10 + 8z - \frac{z^2}{2} - 30}{12} \\
&= \frac{8z - \frac{z^2}{2} - 20}{12} \\
&= \frac{16z - z^2 - 40}{24}
\end{aligned}$$

Summary of CDF:

$$F_Z(z) = \begin{cases} 0 & \text{for } z < 0 \\ \frac{z^2}{24} & \text{for } 0 \leq z \leq 2 \\ \frac{z-1}{6} & \text{for } 2 < z \leq 6 \\ \frac{16z-z^2-40}{24} & \text{for } 6 < z \leq 8 \\ 1 & \text{for } z > 8 \end{cases}$$

The CDF should be sketched as:

- Starts at $(0, 0)$
- Curves upward (parabolic) from $(0, 0)$ to $(2, \frac{1}{6})$
- Linear increase from $(2, \frac{1}{6})$ to $(6, \frac{5}{6})$
- Curves upward (concave down) from $(6, \frac{5}{6})$ to $(8, 1)$
- Constant at 1 for $z > 8$



Question 7

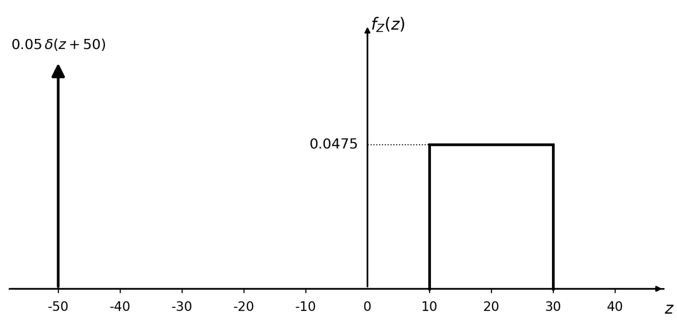
For the sensor that fails with probability 0.05 (error code -50), otherwise reading Z distributed between 10°C and 30°C:

(a) **Mixed PDF of Z:**

The random variable has both a discrete component (the error code) and a continuous component. The PDF is:

$$f_Z(z) = \begin{cases} 0.05 \cdot \delta(z+50), & z = -50 \\ 0.95 \cdot \frac{1}{20} = 0.0475, & 10 \leq z \leq 30 \\ 0, & \text{otherwise} \end{cases}$$

where $\delta(\cdot)$ represents the Dirac delta function at $z = -50$.



(b) **CDF $F_Z(z)$:** The CDF is $F_Z(z) = P(Z \leq z) = \int_{-\infty}^{z} f_Z(t)\,dt$.

Case 1: $z < -50$

$$F_Z(z) = \int_{-\infty}^{z} 0\,dt = 0$$

Case 2: $z = -50$

$$F_Z(-50) = \int_{-\infty}^{-50} 0\,dt + \int_{-50}^{-50} 0.05\delta(t+50)\,dt = 0 + 0.05 = 0.05$$

Case 3: $-50 < z < 10$ The delta function at $t = -50$ is included in the integration range:

$$F_Z(z) = \int_{-\infty}^{z} f_Z(t)\,dt = \int_{-\infty}^{z} 0.05\delta(t+50)\,dt = 0.05$$

Case 4: $10 \le z \le 30$

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{z} f_Z(t)\,dt \\ &= \int_{-\infty}^{10} f_Z(t)\,dt + \int_{10}^{z} f_Z(t)\,dt \\ &= 0.05 + \int_{10}^{z} 0.0475\,dt \\ &= 0.05 + 0.0475[t]_{10}^{z} \\ &= 0.05 + 0.0475(z-10) \end{aligned}$$

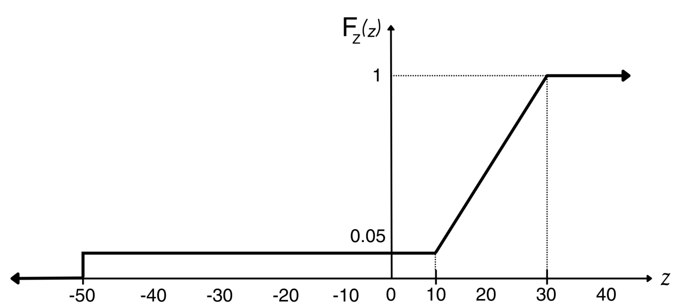
Case 5: $z > 30$

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{30} f_Z(t)\,dt \\ &= 0.05 + \int_{10}^{30} 0.0475\,dt \\ &= 0.05 + 0.0475(30-10) \\ &= 0.05 + 0.95 = 1 \end{aligned}$$

Final CDF:

$$F_Z(z) = \begin{cases} 0, & z < -50 \\ 0.05, & -50 \le z < 10 \\ 0.05 + 0.0475(z-10), & 10 \le z \le 30 \\ 1, & z > 30 \end{cases}$$

Verification: At $z = 30$: $F_Z(30) = 0.05 + 0.0475(30-10) = 0.05 + 0.95 = 1$ ✓



(c) **Probability of error indication:**

The probability that the sensor indicates an error is:

$$P(\text{Error}) = P(Z = -50) = 0.05$$

(d) **Probability that $Z > 25$:**

Using the CDF:

$$\begin{aligned}
P(Z > 25) &= 1 - F_Z(25) \\
&= 1 - [0.05 + 0.0475(25 - 10)] \\
&= 1 - [0.05 + 0.0475 \times 15] \\
&= 1 - [0.05 + 0.7125] \\
&= 1 - 0.7625 = 0.2375
\end{aligned}$$

(e) **Expected sensor reading $E[Z]$:**

Using the definition of expectation for mixed distributions:

$$\begin{aligned}
E[Z] &= \int_{-\infty}^{\infty} z \cdot f_Z(z)\,dz \\
&= (-50) \times 0.05 + \int_{10}^{30} z \cdot 0.95 \cdot \frac{1}{20}\,dz \\
&= -2.5 + 0.0475 \int_{10}^{30} z\,dz \\
&= -2.5 + 0.0475 \left[\frac{z^2}{2}\right]_{10}^{30} \\
&= -2.5 + 0.0475 \left(\frac{30^2 - 10^2}{2}\right) \\
&= -2.5 + 0.0475 \left(\frac{900 - 100}{2}\right) \\
&= -2.5 + 0.0475 \times 400 \\
&= -2.5 + 19 \\
&= 16.5
\end{aligned}$$

Interpretation: The expected sensor reading is 16.5°C, which lies within the normal temperature range of $[10°\mathrm{C}, 30°\mathrm{C}]$. However, it is lower than the mean of 20°C that would be expected if the sensor never failed. The -50 error readings therefore bias the overall mean downward.