## Question 1

(a) Let $X$ denote the number of heads in four tosses. The events $A_k = \{X = k\}$ form a partition of the sample space $S$:

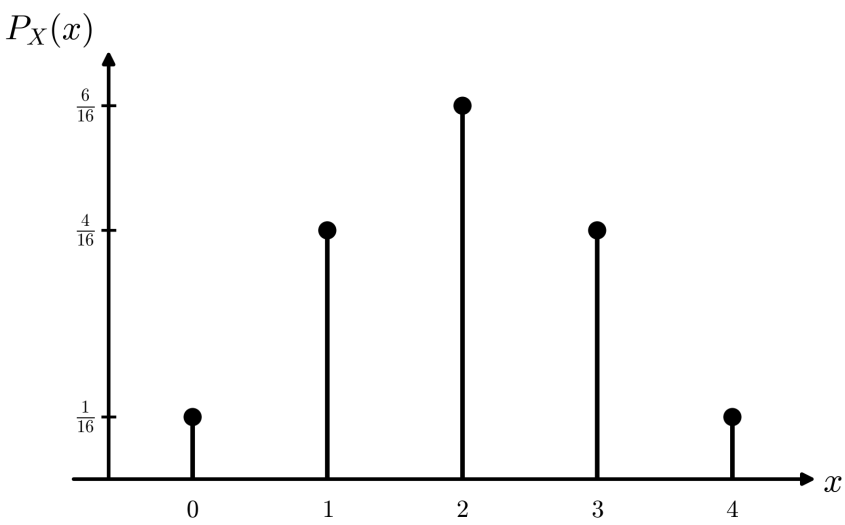
$S$ **(all 16 outcomes)**

| $A_0 = \{X = 0\}$ 0 heads | $A_1 = \{X = 1\}$ 1 head | $A_2 = \{X = 2\}$ 2 heads | $A_3 = \{X = 3\}$ 3 heads | $A_4 = \{X = 4\}$ 4 heads |
|---|---|---|---|---|
| TTTT | HTTT THTT TTHT TTTH | HHTT HTHT HTTH THHT THTH TTHH | HHHT HHTH HTHH THHH | HHHH |

$$S = A_0 \,\dot{\cup}\, A_1 \,\dot{\cup}\, A_2 \,\dot{\cup}\, A_3 \,\dot{\cup}\, A_4, \qquad A_i \cap A_j = \varnothing \text{ for } i \neq j.$$

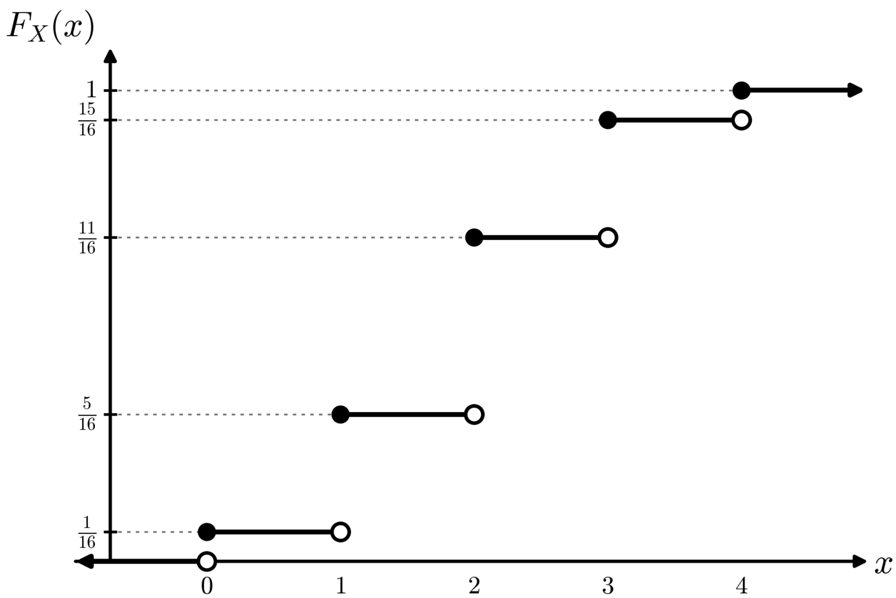
(b) The PMF $\mathrm{P}_X(x)$ is

$$\mathrm{P}_X(x) = \begin{cases} \frac{1}{16} & x = 0 \\ \frac{4}{16} & x = 1 \\ \frac{6}{16} & x = 2 \\ \frac{4}{16} & x = 3 \\ \frac{1}{16} & x = 4 \\ 0 & \text{otherwise} \end{cases}$$



(c) The cumulative probability distribution $F_X(x)$ is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16} & 0 \leq x < 1 \\ \frac{5}{16} & 1 \leq x < 2 \\ \frac{11}{16} & 2 \leq x < 3 \\ \frac{15}{16} & 3 \leq x < 4 \\ 1 & 4 \leq x \end{cases}$$




(d) We can determine $\mathrm{P}(0 < X \leq 2)$ by using $F_X(x)$:

$$\begin{aligned} \mathrm{P}(0 < X \leq 2) &= F_X(2) - F_X(0) \\ &= \frac{11}{16} - \frac{1}{16} \\ &= \frac{10}{16} \\ &= 0.625 \end{aligned}$$

## Question 2

Using the CDF we can determine:

(a)

$$\mathrm{P}(Y < 1) = 0$$

(b)

$$\mathrm{P}(Y \leq 1) = 0.1$$

(c)

$$\mathrm{P}(Y \geq 2) = 1 - \mathrm{P}(Y < 2) = 1 - F_Y(1) = 1 - 0.1 = 0.9$$

(d)

$$\mathrm{P}(Y = 3) = F_Y(3) - F_Y(3^-) = 0.6 - 0.4 = 0.2$$

(e)

$$\mathrm{P}(Y > 3) = 1 - \mathrm{P}(Y \leq 3) = 1 - 0.6 = 0.4$$

(f)

$$\mathrm{P}_Y(y) = \begin{cases} 0.1 & y = 1 \\ 0.3 & y = 2 \\ 0.2 & y = 3 \\ 0.3 & y = 4 \\ 0.1 & y = 5 \\ 0 & \text{otherwise} \end{cases}$$

## Question 3

The transformation gives

$$X = -1 \mapsto Y = 4, \qquad X = 0 \mapsto Y = 1, \qquad X = 2 \mapsto Y = 1.$$

(a) The two probabilities that map to $Y = 1$ must be added:

$$P_Y(y) = \begin{cases} 0.8, & y = 1, \\ 0.2, & y = 4, \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$\begin{aligned} \mathrm{E}[Y] &= 1(0.8) + 4(0.2) = 1.6, \\ \mathrm{E}[Y^2] &= 1^2(0.8) + 4^2(0.2) = 4, \\ \mathrm{Var}[Y] &= 4 - (1.6)^2 = 1.44. \end{aligned}$$

(c) The event $Y > 2$ occurs only when $Y = 4$. Therefore

$$P_A(a) = \begin{cases} 0.8, & a = 0, \\ 0.2, & a = 1, \\ 0, & \text{otherwise.} \end{cases}$$

## Question 4

### Step 1: Define relative probabilities

We are told the scoring zones relate as follows:

$$\begin{aligned} \mathrm{P}_S(4) &= p \quad \text{(Zone A)} \\ \mathrm{P}_S(3) &= 2p \quad \text{(Zone B)} \\ \mathrm{P}_S(2) &= 4p \quad \text{(Zone C)} \\ \mathrm{P}_S(1) &= 8p \quad \text{(Zone D)} \end{aligned}$$

Total probability of hitting the target is 0.3:

$$p + 2p + 4p + 8p = 15p = 0.3 \Rightarrow p = \frac{0.3}{15} = 0.02$$

### Step 2: Compute full PMF of $S$

Now that we have $p$, we compute:

$$\begin{aligned} &\mathrm{P}_S(4) = p = 0.02 \\ &\mathrm{P}_S(3) = 2p = 0.04 \\ &\mathrm{P}_S(2) = 4p = 0.08 \\ &\mathrm{P}_S(1) = 8p = 0.16 \\ &\mathrm{P}_S(0) = 1 - 0.3 = 0.7 \quad \text{(missed target)} \end{aligned}$$

### Final Answer: PMF of $S$

$$\mathrm{P}_S(s) = \begin{cases} 0.7 & s = 0 \\ 0.16 & s = 1 \\ 0.08 & s = 2 \\ 0.04 & s = 3 \\ 0.02 & s = 4 \\ 0 & \text{otherwise} \end{cases}$$

### Tip for students:

If the probabilities follow a consistent ratio, define the smallest as $p$ and scale up. Then use the known total probability to solve for $p$. This approach avoids brute force and ensures you get a normalized PMF.

# Question 5

## Step 1: Define the Random Variable

Let $X$ be the time (in hours) it takes Sarah to reach her friend's house.

- **Cycle:** $\frac{2.4}{12} = 0.2$ hours, with $\mathrm{P}(X = 0.2) = 0.3$
- **Jog:** $\frac{2.4}{6} = 0.4$ hours, with $\mathrm{P}(X = 0.4) = 0.5$
- **Walk:** $\frac{2.4}{4} = 0.6$ hours, with unknown probability $p$

## Step 2: Find the Missing Probability

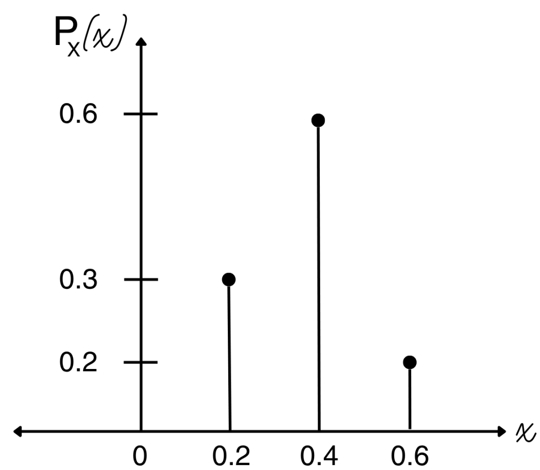
Because the total probability must equal 1:

$$0.3 + 0.5 + p = 1 \Rightarrow p = 0.2$$

So, $\mathrm{P}(X = 0.6) = 0.2$

## Step 3: PMF of $X$

$$\mathrm{P}_X(x) = \begin{cases} 0.3 & \text{if } x = 0.2 \\ 0.5 & \text{if } x = 0.4 \\ 0.2 & \text{if } x = 0.6 \\ 0 & \text{otherwise} \end{cases}$$




## Step 4: Expected Value

$$\mathrm{E}[X] = (0.2)(0.3) + (0.4)(0.5) + (0.6)(0.2) = 0.06 + 0.20 + 0.12 = 0.38 \text{ hours}$$

Convert to minutes:

$$0.38 \times 60 = 22.8 \text{ minutes}$$

## Final Answer:

On average, it takes Sarah **22.8 minutes** to reach her friend's house.

### Important Note:

It is **incorrect** to first calculate the average speed E$[V]$ and then use it to estimate time, i.e.:

$$\mathrm{E}[V] = 12 \cdot 0.3 + 6 \cdot 0.5 + 4 \cdot 0.2 = 3.6 + 3.0 + 0.8 = 7.4$$
$$T = \frac{2.4}{\mathrm{E}[V]} = \frac{2.4}{7.4} \approx 0.324 \text{ hours}$$

This is **not** equal to the correct expected time of $\mathrm{E}[X] = 0.38$ hours.
This error occurs because the transformation $g(X) = \frac{2.4}{X}$ is **non-linear**, and thus:

$$\mathrm{E}[g(X)] \neq g(\mathrm{E}[X])$$

Always apply the expectation to the correct random variable—time in this case—not its inverse (speed).

## Question 6

(a) **Standard Deviation of $X$**

The standard deviation is a measure of how much the values of a random variable deviate from the mean. To compute this, we first calculate the mean $\mu_X$:

$$\begin{aligned}\mu_X &= \frac{1}{6}(12+17+20+45+50+57)\\ &= \frac{201}{6} = 33.5\end{aligned}$$

Next, we calculate the variance $\sigma_X^2 = \mathrm{E}[(X-\mu_X)^2]$:

$$\begin{aligned}\sigma_X^2 = \mathrm{E}[(X-\mu_X)^2] &= \sum_{\forall x} \mathrm{P}_X(x)(x-\mu_X)^2\\ &= \frac{1}{6}\big[(12-33.5)^2 + (17-33.5)^2 + (20-33.5)^2\\ &\quad +(45-33.5)^2 + (50-33.5)^2 + (57-33.5)^2\big]\\ &= \frac{1}{6}(1873.5) = 312.25\end{aligned}$$

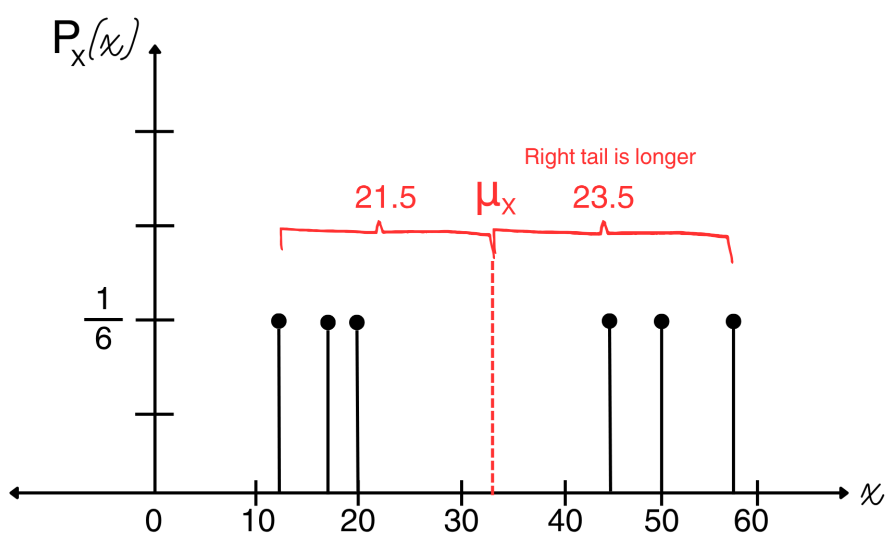
Then, the standard deviation is:

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{312.25} = 17.67$$

**Interpretation:** The standard deviation of 17.67 indicates that, on average, the values of $X$ deviate from the mean by about 18 units. It gives us a sense of how "spread out" the values are.

(b) **Coefficient of Skewness of $X$**

The coefficient of skewness measures the asymmetry of the distribution of values. A positive value indicates a longer tail on the right.



First, compute the third central moment $\mu_3 = \mathrm{E}[(X - \mu_X)^3]$:

$$\begin{aligned}
\mu_3 = E[(X - \mu_X)^3] &= \sum_{\forall x} \mathrm{P}_X(x)(x - \mu_X)^3 \\
&= \frac{1}{6}\big[(12 - 33.5)^3 + (17 - 33.5)^3 + (20 - 33.5)^3 \\
&\quad + (45 - 33.5)^3 + (50 - 33.5)^3 + (57 - 33.5)^3\big] \\
&= \frac{1}{6}(2100) = 350
\end{aligned}$$

Now use the standard deviation from part (a) to calculate the coefficient of skewness:

$$\gamma = \frac{\mu_3}{\sigma_X^3} = \frac{350}{17.67^3} = \frac{350}{5517.08} = 0.0634$$

**Interpretation:** The coefficient of skewness is approximately 0.0634, which is a small positive value. This suggests the distribution is slightly skewed to the right—there is a longer tail on the higher value side.

## Question 7

(a) **Voltage Values and PMF**

The possible voltage values are:

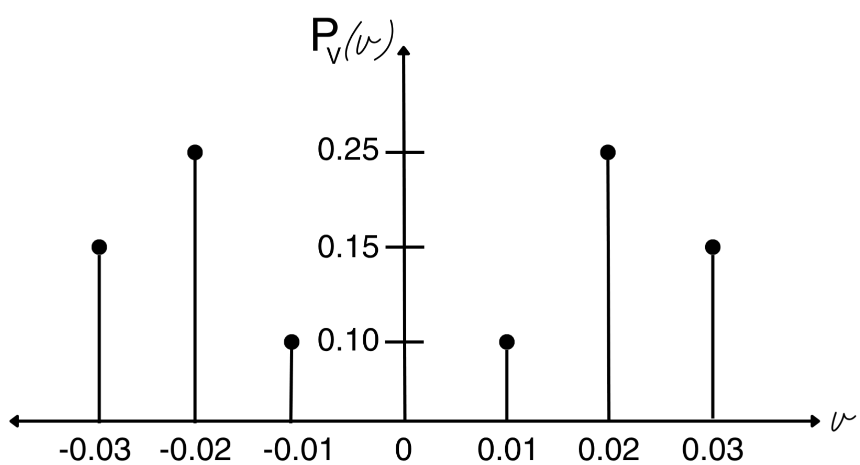
$$v = \{-0.03, -0.02, -0.01, 0.01, 0.02, 0.03\}$$

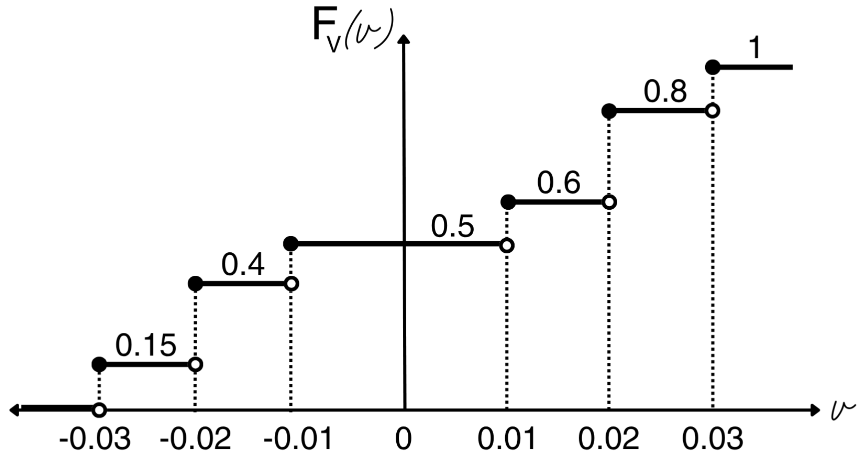
These result from three types of particles:

- $\alpha$ particles: $\pm 0.01$ with probability 0.1 total (i.e., 0.05 each, or 0.1 split equally over $\pm 0.01$),
- $\beta$ particles: $\pm 0.02$ with total probability 0.25,
- $\gamma$ particles: $\pm 0.03$ with total probability 0.15.

Since each voltage magnitude has a positive and negative form with equal likelihood:

$$\mathrm{P}_V(v) = \begin{cases} 0.15 & v = -0.03 \\ 0.25 & v = -0.02 \\ 0.10 & v = -0.01 \\ 0.10 & v = 0.01 \\ 0.25 & v = 0.02 \\ 0.15 & v = 0.03 \\ 0 & \text{otherwise} \end{cases}$$






(b) **Mean and Variance of V**

*Mean:*

$$\begin{aligned}\mu_V &= (-0.03)(0.15) + (-0.02)(0.25) + (-0.01)(0.10) \\ &\quad + (0.01)(0.10) + (0.02)(0.25) + (0.03)(0.15) \\ &= 0\end{aligned}$$

*Variance:*

$$\begin{aligned}\sigma_V^2 &= \sum v^2 \mathrm{P}_V(v) - \mu_V^2 \\ &= (-0.03)^2(0.15) + (-0.02)^2(0.25) + (-0.01)^2(0.10) \\ &\quad + (0.01)^2(0.10) + (0.02)^2(0.25) + (0.03)^2(0.15) \\ &= 0.00049 = 4.9 \times 10^{-4}\end{aligned}$$

(c) **Effect of Amplification: Output Variance**

For a zero mean signal, the input variance is:

$$\sigma_V^2 = \sum V^2 \mathrm{P}_V(v)$$

The output signal can be expressed as $f(V) = 100V$, the output variance is therefor:

$$\sigma_{f(V)}^2 = \sum f(V)^2 \mathrm{P}_V(v) = (10000) \sum V^2 \mathrm{P}_V(v)$$

The output variance is the input variance multiplied by 10000.

Thus, the output variance is 4.9.

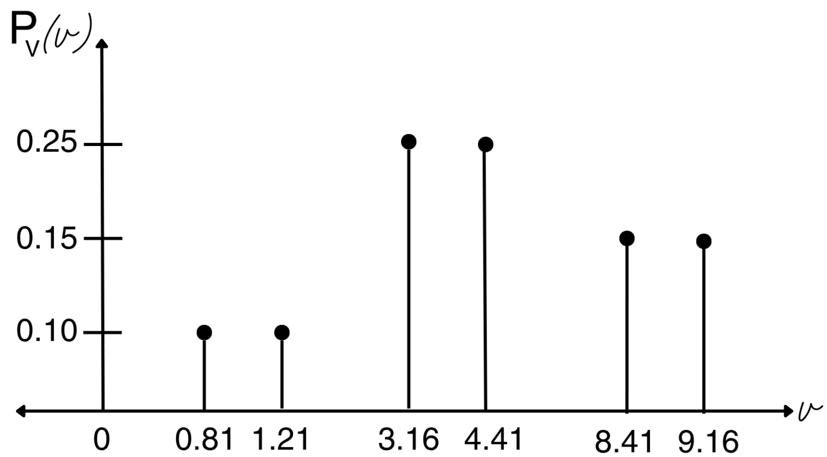
(d) **Squaring Transformation and PMF**

Each input value $v$ is transformed as $(0.1 + 100v)^2$. Compute each:

$$\begin{aligned}(0.1 - 3)^2 &= 8.41 \\ (0.1 - 2)^2 &= 3.61 \\ (0.1 - 1)^2 &= 0.81 \\ (0.1 + 1)^2 &= 1.21 \\ (0.1 + 2)^2 &= 4.41 \\ (0.1 + 3)^2 &= 9.61\end{aligned}$$

The transformed probability mass function is:

$$\mathrm{P}_V(v) = \begin{cases} 0.10 & v = 0.81 \\ 0.10 & v = 1.21 \\ 0.15 & v = 3.61 \\ 0.25 & v = 4.41 \\ 0.25 & v = 8.41 \\ 0.15 & v = 9.61 \\ 0 & \text{otherwise} \end{cases}$$




(e) **Physical Interpretation of Output System**

The combination of amplifier, squaring circuit, and low-pass filter (LPF) behaves as a hardware-based variance estimator. Since:

$$f(V) = (0.1 + 100v)^2$$

the expected output of the LPF is:

$$\begin{aligned} \mathbb{E}[f(V)] &= \sum f(v)\mathrm{P}_V(v) \\ &= (8.41)(0.15) + (3.61)(0.25) + (0.81)(0.10) \\ &\quad + (1.21)(0.10) + (4.41)(0.25) + (9.61)(0.15) \\ &= 4.91 \end{aligned}$$

(f) **Comparison with Theoretical Expectation**

From part (c), the expected output is:

$$\sigma^2_{f(V)} = 10^4 \cdot \sigma^2_V = 10^4 \cdot 0.00049 = 4.9$$

However, the actual output is:

$$\mathrm{E}[f(V)] = 4.91$$

The small discrepancy (0.01) arises due to a non-ideal amplifier with a slight output offset (e.g., the constant 0.1 in the transformation), causing bias in the measurement. This makes the system a **biased estimator**, a concept which will be explored further later in the module.