Question 1

(a) We use the law of total probability:

$$\begin{aligned}\mathrm{P}(Ball_2 = Black) = {} & \mathrm{P}(Ball_2 = Black|Ball_1 = Yellow)\mathrm{P}(Ball_1 = Yellow) \\ & + \mathrm{P}(Ball_2 = Black|Ball_1 = White)\mathrm{P}(Ball_1 = White) \\ & + \mathrm{P}(Ball_2 = Black|Ball_1 = Black)\mathrm{P}(Ball_1 = Black)\end{aligned}$$

Total number of balls initially: $4 + 7 + 5 = 16$ After drawing one ball: 15 left.

$$\mathrm{P}(Ball_1 = Yellow) = \frac{4}{16} \tag{1}$$
$$\mathrm{P}(Ball_1 = White) = \frac{5}{16} \tag{2}$$
$$\mathrm{P}(Ball_1 = Black) = \frac{7}{16} \tag{3}$$
$$\mathrm{P}(Ball_2 = Black|Ball_1 = Yellow) = \frac{7}{15} \tag{4}$$
$$\mathrm{P}(Ball_2 = Black|Ball_1 = White) = \frac{7}{15} \tag{5}$$
$$\mathrm{P}(Ball_2 = Black|Ball_1 = Black) = \frac{6}{15} \tag{6}$$

Substituting into the total probability formula:

$$\begin{aligned}\mathrm{P}(Ball_2 = Black) &= \frac{7}{15}\cdot\frac{4}{16} + \frac{7}{15}\cdot\frac{5}{16} + \frac{6}{15}\cdot\frac{7}{16} \\ &= \frac{28 + 35 + 42}{240} = \frac{105}{240} = 0.4375\end{aligned}$$

(b) Now use Bayes' Rule:

$$\begin{aligned}\mathrm{P}(Ball_1 = Yellow|Ball_2 = Black) &= \frac{\mathrm{P}(Ball_2 = Black|Ball_1 = Yellow)\cdot\mathrm{P}(Ball_1 = Yellow)}{\mathrm{P}(Ball_2 = Black)} \\ &= \frac{\frac{7}{15}\cdot\frac{4}{16}}{0.4375} \\ &= \frac{28}{240} \div 0.4375 = \frac{28}{240}\cdot\frac{1}{0.4375} \\ &= \frac{28}{240}\cdot\frac{16}{7} = \frac{28\cdot 16}{240\cdot 7} = \frac{448}{1680} = \frac{28}{105}\end{aligned}$$

Final answer:

$$\mathrm{P}(Ball_1 = Yellow|Ball_2 = Black) = \frac{4}{15}$$

Question 2

Let the physical doors be D_1, D_2, and D_3, and assume that the participant initially chooses D_1. To distinguish doors from events, define C_i as the event that the car is behind door D_i, for $i \in \{1, 2, 3\}$. Define

$$H_{D2} := \{\text{the host opens door } D_2\}.$$

After observing H_{D2}, staying means keeping D_1, whereas switching means choosing the only other unopened door, D_3. We therefore compare $\mathrm{P}(C_1 \mid H_{D2})$ and $\mathrm{P}(C_3 \mid H_{D2})$.
The host never opens the participant's chosen door or the door containing the car. If both unchosen doors contain goats, the stated "no preference" rule means that the host chooses each of them with probability 1/2.

Step 1: Define the prior probabilities

Before any door is opened, the car is equally likely to be behind each door:

$$\mathrm{P}(C_1) = \mathrm{P}(C_2) = \mathrm{P}(C_3) = \frac{1}{3}.$$

Step 2: Determine the likelihood of H_{D2} in each case

$$\begin{aligned}
\mathrm{P}(H_{D2} \mid C_1) &= \frac{1}{2} && \text{(both } D_2 \text{ and } D_3 \text{ contain goats)},\\
\mathrm{P}(H_{D2} \mid C_2) &= 0 && \text{(the host cannot reveal the car behind } D_2\text{)},\\
\mathrm{P}(H_{D2} \mid C_3) &= 1 && \text{(the host is forced to open } D_2\text{)}.
\end{aligned}$$

Step 3: Calculate $\mathrm{P}(H_{D2})$

Using the law of total probability over the three possible car locations,

$$\begin{aligned}
\mathrm{P}(H_{D2}) &= \sum_{i=1}^{3} \mathrm{P}(H_{D2} \mid C_i)\mathrm{P}(C_i)\\
&= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) = \frac{1}{2}.
\end{aligned}$$

Step 4: Compare staying and switching

Bayes' theorem gives the posterior probability for each possible car location. For the initially chosen door D_1 and the remaining closed door D_3,

$$\begin{aligned}
\mathrm{P}(C_1 \mid H_{D2}) &= \frac{\mathrm{P}(H_{D2} \mid C_1)\mathrm{P}(C_1)}{\mathrm{P}(H_{D2})} = \frac{(1/2)(1/3)}{1/2} = \frac{1}{3},\\
\mathrm{P}(C_3 \mid H_{D2}) &= \frac{\mathrm{P}(H_{D2} \mid C_3)\mathrm{P}(C_3)}{\mathrm{P}(H_{D2})} = \frac{(1)(1/3)}{1/2} = \frac{2}{3}.
\end{aligned}$$

Also, $\mathrm{P}(C_2 \mid H_{D2}) = 0$, because the opened door contains a goat. The two closed doors are therefore *not* equally likely: observing H_{D2} is twice as likely when the car is behind D_3 as when it is behind D_1.

Step 5: Conclude the optimal strategy

Staying with D_1 wins with probability 1/3, whereas switching to D_3 wins with probability 2/3. Equivalently, staying wins only when the initial choice was correct, while switching wins whenever it was wrong. By symmetry, the same conclusion holds if the host opens D_3 instead. Under the stated host rules, the participant should **switch**.

Question 3

(a) We are given that 4 tiles are randomly drawn from a set of 60 uniquely numbered tiles (1 to 60). We are to calculate the probability of correctly guessing *all four* numbers that are drawn. Since order does not matter, this is a combinations problem:

$$\begin{aligned}\text{Total number of 4-tile combinations} &= \binom{60}{4} = \frac{60!}{(60-4)! \cdot 4!} \\ &= \frac{60 \times 59 \times 58 \times 57}{4 \times 3 \times 2 \times 1} = \frac{11{,}703{,}240}{24} = 487{,}635\end{aligned}$$

Since there is only **one correct combination**, the probability of guessing it exactly is:

$$\text{P(guess all 4 correctly)} = \frac{1}{487{,}635}$$

(b) Now we calculate the probability of guessing *exactly two* of the four numbers correctly.

Step 1: Choose 2 correct numbers from the 4 drawn:

$$\binom{4}{2} = 6$$

Step 2: The remaining 2 guesses must be incorrect — these must come from the 56 numbers *not drawn* (i.e., $60 - 4 = 56$):

$$\binom{56}{2} = \frac{56 \times 55}{2} = 1{,}540$$

Step 3: Total number of favourable outcomes (exactly 2 correct guesses):

$$6 \times 1{,}540 = 9{,}240$$

Step 4: Divide by the total number of possible 4-number guesses:

$$\text{P(exactly 2 correct)} = \frac{9{,}240}{487{,}635} \approx 0.01895$$

Final Answer:

$$\text{P(guess exactly 2 correctly)} = \frac{9{,}240}{487{,}635} \approx 1.89\%$$

Question 4

Each position is labelled and has different responsibilities, so it matters which technician receives which job: swapping two technicians changes the roster. This is an ordered assignment (a permutation), not an unordered selection (a combination). For r distinct positions filled from n people without repetition, use

$$P_r^n = \frac{n!}{(n-r)!}.$$

(a) Assign the three on-site roles

$$P_3^8 = \frac{8!}{(8-3)!} = 8 \times 7 \times 6 = \boxed{336}.$$

The roles are different, and no technician may be used twice, so order matters.

(b) Add the first and second reserves

Use the multiplication principle. First assign the three distinct on-site roles. Five technicians then remain, from whom the distinct first- and second-reserve positions must be filled. Therefore,

$$N = P_3^8\, P_2^5 = \left(\frac{8!}{5!}\right)\left(\frac{5!}{3!}\right) = 336(20) = \boxed{6720}.$$

Equivalently, all five labelled positions can be assigned at once: $N = P_5^8 = 8!/3! = 6720$.

Question 5

Definitions and given data.
Let D denote that a module is actually defective, let $\overline{D}$ denote that it is actually non-defective, and let R_+ and R_- denote positive and negative inspection results. Let I denote that the inspection result is incorrect (so $\overline{I}$ denotes a correct result). The four possible classification blocks are

Inspection result	D: actually defective	$\overline{D}$: actually non-defective
R_+: positive	$\mathrm{TP} = R_+ \cap D$	**FP** $= R_+ \cap \overline{D}$ **(in I)**
R_-: negative	**FN** $= R_- \cap D$ **(in I)**	$\mathrm{TN} = R_- \cap \overline{D}$

The two bold, off-diagonal blocks form $I = \mathrm{FP} \cup \mathrm{FN} = (R_+ \cap \overline{D}) \cup (R_- \cap D)$; the diagonal blocks form $\overline{I} = \mathrm{TP} \cup \mathrm{TN}$. The given probabilities are

$$\mathrm{P}(I) = 0.08, \quad \mathrm{P}(R_-) = \frac{900}{1200} = 0.75, \quad \mathrm{P}(R_+) = \frac{300}{1200} = 0.25, \quad \mathrm{P}(I \mid R_+) = 0.15.$$

(a) Probability of a defect given a negative result.
Partitioning the incorrect results according to the observed result gives the law of total probability

$$\mathrm{P}(I) = \mathrm{P}(I \mid R_-)\mathrm{P}(R_-) + \mathrm{P}(I \mid R_+)\mathrm{P}(R_+).$$

Hence,

$$\begin{aligned}\mathrm{P}(I \mid R_-) &= \frac{\mathrm{P}(I) - \mathrm{P}(I \mid R_+)\mathrm{P}(R_+)}{\mathrm{P}(R_-)} \\ &= \frac{0.08 - (0.15)(0.25)}{0.75} = \frac{17}{300} \approx 0.0567.\end{aligned}$$

A negative result is incorrect exactly when the module is actually defective. Therefore,

$$\boxed{\mathrm{P}(D \mid R_-) = \frac{17}{300} \approx 0.0567.}$$

(b) Compare false-positive and false-negative counts.

$$\begin{aligned}N_{FN} &= 900\left(\frac{17}{300}\right) = 51, \\ N_{FP} &= 300(0.15) = 45.\end{aligned}$$

There are therefore **more false-negative results** (51) than false-positive results (45).

(c) Probability that a module is actually non-defective.
A non-defective module either receives a correct negative result or an incorrect positive result. Therefore,

$$\begin{aligned}\mathrm{P}(\overline{D}) &= \mathrm{P}(\overline{I} \mid R_-)\mathrm{P}(R_-) + \mathrm{P}(I \mid R_+)\mathrm{P}(R_+) \\ &= \left(1 - \frac{17}{300}\right)(0.75) + (0.15)(0.25) \\ &= 0.7075 + 0.0375 = 0.745.\end{aligned}$$

$$\boxed{\mathrm{P}(\overline{D}) = 0.745.}$$

(d) Expected number of defective modules.

$$\begin{aligned}\mathrm{P}(D) &= 1 - \mathrm{P}(\overline{D}) = 1 - 0.745 = 0.255, \\ N_D &= 1200\,\mathrm{P}(D) = 1200(0.255) = 306.\end{aligned}$$

$$\boxed{\text{306 modules are expected to be actually defective.}}$$

(e) Probability of a negative result given an error.
By Bayes' theorem,

$$\begin{aligned}\mathrm{P}(R_- \mid I) &= \frac{\mathrm{P}(I \mid R_-)\mathrm{P}(R_-)}{\mathrm{P}(I)} \\ &= \frac{(17/300)(0.75)}{0.08} = \frac{0.0425}{0.08} = 0.53125.\end{aligned}$$

$$\boxed{\mathrm{P}(R_- \mid I) = 0.53125 \approx 53.1\%.}$$

Question 6

Step 1: Define events

Let:

$$\begin{aligned} M &= \{\text{the data session contains a malicious intrusion}\}, \\ \overline{M} &= \{\text{the data session is legitimate}\}, \\ A &= \{\text{the monitoring system raises an alert}\}. \end{aligned}$$

Step 2: Known probabilities

From the question:

$$\begin{aligned} \mathrm{P}(M) &= 0.02, & \mathrm{P}(\overline{M}) &= 1 - \mathrm{P}(M) = 0.98, \\ \mathrm{P}(A \mid M) &= 0.92, & \mathrm{P}(A \mid \overline{M}) &= 0.07. \end{aligned}$$

Step 3: Goal

We want:

$$\mathrm{P}(M \mid A) = \frac{\mathrm{P}(A \mid M)\mathrm{P}(M)}{\mathrm{P}(A)}.$$

Step 4: Compute $\mathrm{P}(A)$ using total probability

$$\begin{aligned} \mathrm{P}(A) &= \mathrm{P}(A \mid M)\mathrm{P}(M) + \mathrm{P}(A \mid \overline{M})\mathrm{P}(\overline{M}) \\ &= (0.92)(0.02) + (0.07)(0.98) \\ &= 0.0184 + 0.0686 = 0.087. \end{aligned}$$

Step 5: Compute $\mathrm{P}(M \mid A)$

$$\begin{aligned} \mathrm{P}(M \mid A) &= \frac{\mathrm{P}(A \mid M)\mathrm{P}(M)}{\mathrm{P}(A)} \\ &= \frac{(0.92)(0.02)}{0.087} = \frac{0.0184}{0.087} \approx 0.2115. \end{aligned}$$

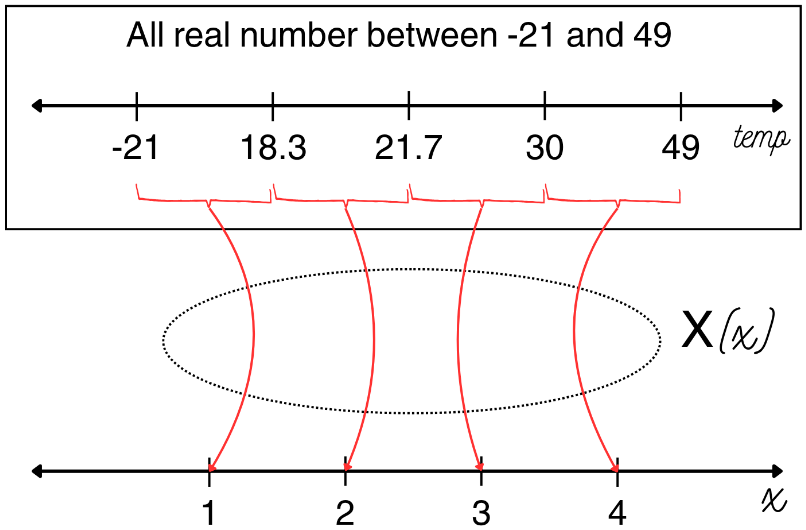
Step 6: Final Answer

$$\boxed{\mathrm{P}(M \mid A) \approx 21.1\%.}$$

Although the monitor detects most malicious sessions, malicious sessions are rare. False alerts from the much larger legitimate group therefore account for many alerts. Given an alert, the probability that the session is actually malicious is approximately **21.1%**.

Question 7

(a) Sample Space Mapping



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(b) Probability Mass Function (PMF) and Graph

The temperature T is uniformly distributed over $[-21, 49]$, giving a total range:

$$\text{Range} = 49 - (-21) = 70$$

For each thermostat setting X, compute:

$$\mathrm{P}(X = 1) = \frac{18.3 - (-21)}{70} = \frac{39.3}{70} \approx 0.5614$$

$$\mathrm{P}(X = 2) = \frac{21.7 - 18.3}{70} = \frac{3.4}{70} \approx 0.0486$$

$$\mathrm{P}(X = 3) = \frac{30 - 21.7}{70} = \frac{8.3}{70} \approx 0.1186$$

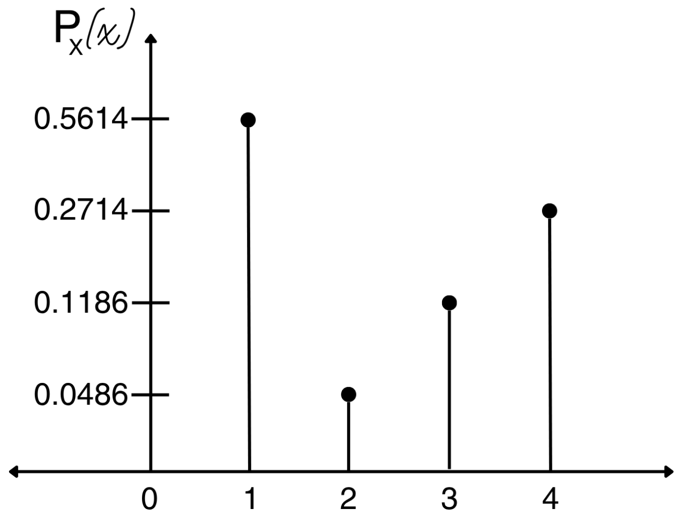
$$\mathrm{P}(X = 4) = \frac{49 - 30}{70} = \frac{19}{70} \approx 0.2714$$

Thus, the probability mass function $P_X(x)$ can be expressed as:

$$\mathrm{P}_X(x) = \begin{cases} 0.5614 & x = 1 \\ 0.0486 & x = 2 \\ 0.1186 & x = 3 \\ 0.2714 & x = 4 \\ 0 & \text{otherwise} \end{cases}$$

Check:

$$\mathrm{P}(X = 1) + \mathrm{P}(X = 2) + \mathrm{P}(X = 3) + \mathrm{P}(X = 4) = 0.5614 + 0.0486 + 0.1186 + 0.2714 = 1$$



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(c) Probability Calculation $\mathrm{P}(1 < X \leq 3)$

$$\mathrm{P}(1 < X \leq 3) = \mathrm{P}(X = 2) + \mathrm{P}(X = 3)$$

$$\mathrm{P}(1 < X \leq 3) = 0.0486 + 0.1186 = 0.1672$$

$$\boxed{\mathrm{P}(1 < X \leq 3) \approx 0.1672}$$