Question 1

(a)

$$A \cap C = \{1\}$$

Channel 1 is both high-data-rate and backed up.

(b)

$$A \cup C = \{1, 2, 3, 4, 5\}$$

$$\mathrm{P}(A \cup C) = \frac{n(A \cup C)}{n(S)} = \frac{5}{6}$$

Thus five of the six possible scans provide at least one performance or resilience benefit.

(c)

$$B = \{2, 5, 6\}$$

$$\overline{B} = \{1, 3, 4\}$$

$$n(\overline{B} \cup C) = n(\overline{B}) + n(C) - n(\overline{B} \cap C)$$

$$= 3 + 4 - 2 = 5$$

$$\mathrm{P}(\overline{B} \cup C) = \frac{n(\overline{B} \cup C)}{n(S)} = \frac{5}{6}$$

Therefore traffic can be carried immediately for five of the six possible channel selections. Channel 6 is the only channel whose primary route is blocked and which has no backup route.

(d)

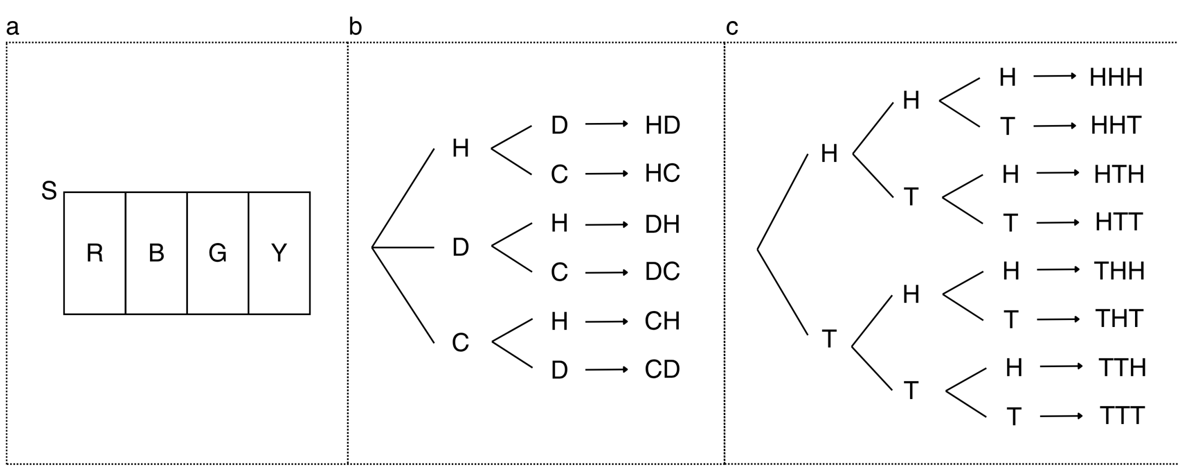
$$\overline{B \cup C} = \overline{B} \cap \overline{C}$$

$$\overline{B} = \{1, 3, 4\}, \quad \overline{C} = \{3, 6\}$$

$$\overline{B \cup C} = \{3\}$$

Channel 3 is currently usable because its primary route is not blocked, but it has no backup route. It is therefore less resilient to a future primary-route failure.

Question 2



(a) The sample space consists of all possible colors the spinner can land on:

$$S = \{\text{red}, \text{blue}, \text{green}, \text{yellow}\}$$

Since the spinner is equally divided into four sections, the outcomes are equiprobable. Therefore, the probability of each outcome is:

$$\text{P(red)} = \text{P(blue)} = \text{P(green)} = \text{P(yellow)} = \frac{1}{4}$$

(b) The sample space consists of all possible ordered pairs of cards drawn without replacement:

$$S = \{(H, D), (H, C), (D, H), (D, C), (C, H), (C, D)\}$$

Since there are 3 cards and the draws are without replacement, there are $3 \times 2 = 6$ possible ordered outcomes. Each has equal probability:

$$\text{P(each pair)} = \frac{1}{6}$$

(c) The sample space consists of the possible number of tails in three coin tosses:

$$S = \{0, 1, 2, 3\}$$

From the full set of 3-coin toss outcomes:

$$\{\text{hhh}, \text{hht}, \text{hth}, \text{thh}, \text{htt}, \text{tht}, \text{tth}, \text{ttt}\}$$

we can group by number of tails:

- 0 tails: {hhh}: $\text{P}(nT = 0) = \frac{1}{8}$
- 1 tail: {hht, hth, thh}: $\text{P}(nT = 1) = \frac{3}{8}$
- 2 tails: {htt, tht, tth}: $\text{P}(nT = 2) = \frac{3}{8}$
- 3 tails: {ttt}: $\text{P}(nT = 3) = \frac{1}{8}$

Question 3

We know, thanks to De Morgan's law for two sets, that:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \tag{1}$$

With that, we can write

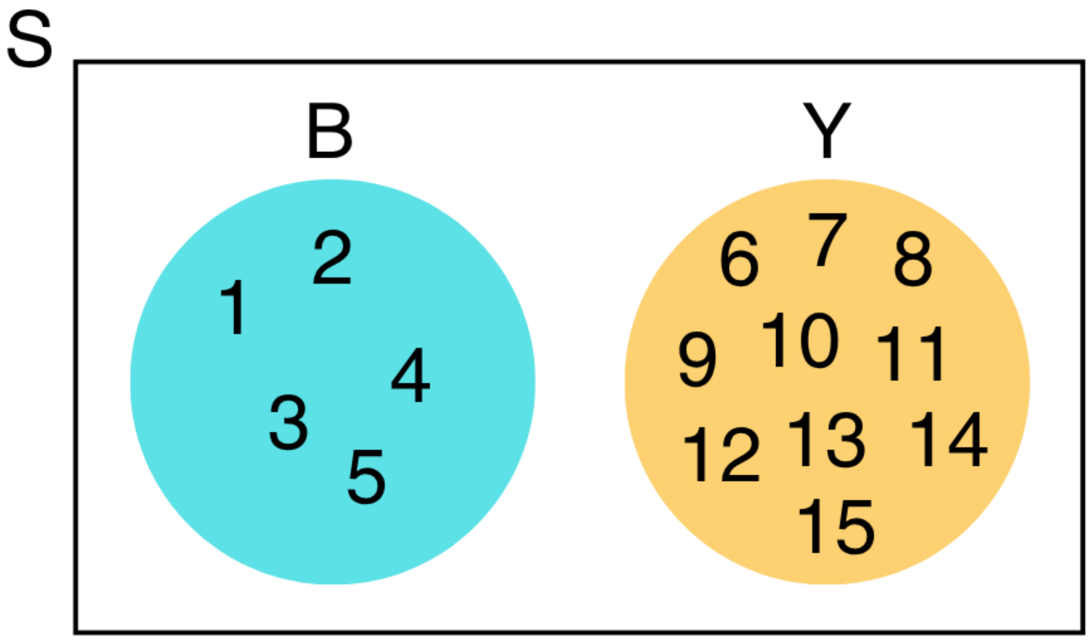
$$\begin{aligned}
\overline{A \cup B \cup C} &= \overline{(A \cup B) \cup C} && \text{(Associative)} \\
&= \overline{A \cup B} \cap \overline{C} && \text{(De Morgan's law)}
\end{aligned}$$

Using eq. 1 we can modify the above equation to be

$$\begin{aligned}
\overline{A \cup B} \cap \overline{C} &= (\overline{A} \cap \overline{B}) \cap \overline{C} \\
&= \overline{A} \cap \overline{B} \cap \overline{C} && \text{(Associative)}
\end{aligned}$$

Question 4

(a) The Venn diagram below shows the two requested events: B, drawing a blue button, and Y, drawing a yellow button.



(b) Let B be the event where a blue button is removed.

$$\mathrm{P}(B) = \frac{n(B)}{n(S)} = \frac{5}{15} = \frac{1}{3}$$

(c) Let event Y be the event where a yellow button is removed, and event O the event where the button number is odd.

First we determine:

$$\begin{aligned} Y &= \{\text{buttons numbered 6 to 15}\} \Rightarrow n(Y) = 10 \\ O &= \{\text{odd-numbered buttons in } S\} = \{1, 3, 5, 7, 9, 11, 13, 15\} \Rightarrow n(O) = 8 \\ Y \cap O &= \{\text{odd yellow buttons}\} = \{7, 9, 11, 13, 15\} \Rightarrow n(Y \cap O) = 5 \end{aligned}$$

Now compute the probability:

$$\mathrm{P}(Y \cap O) = \frac{n(Y \cap O)}{n(S)} = \frac{5}{15} = \frac{1}{3}$$

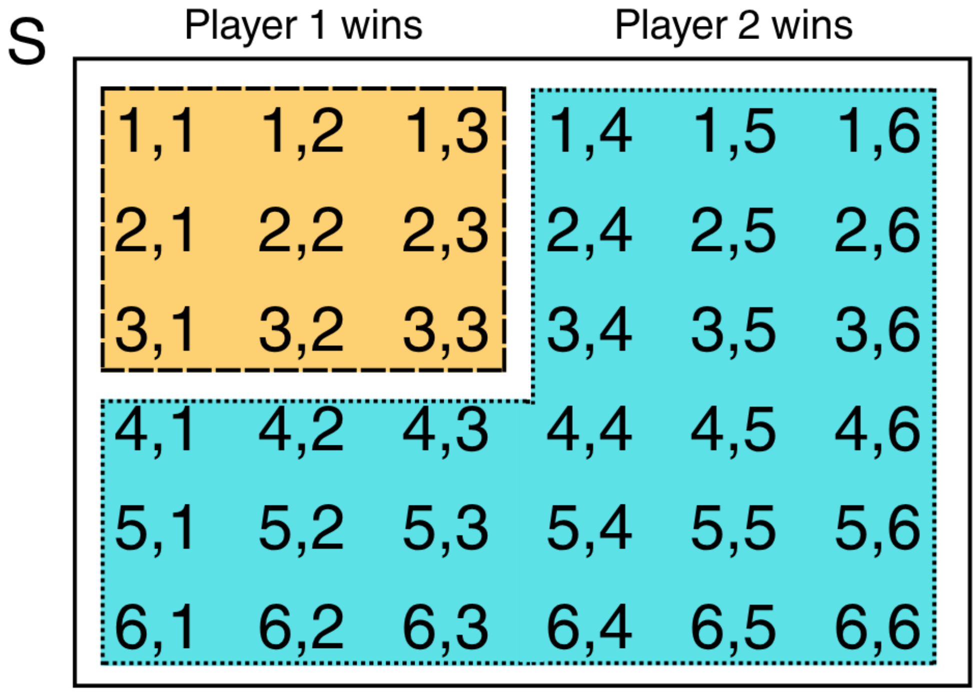
(d) We removed one blue button from the jar, so 14 buttons remain, of which four are blue. Thus the probability that the second button is also blue is:

$$\mathrm{P}(\text{Button}_2 = B \mid \text{Button}_1 = B) = \frac{n(B)}{n(S)} = \frac{4}{14} = \frac{2}{7}$$

Question 5

(a) If there were only one die, then it would be equal, since there are six possible outcomes and three of them lead to player one winning. When there are two dice, the amount of possible outcomes becomes 36, of which very few have a 3 or less as the highest dice roll.

(b) The sample space for the experiment is displayed below:



(c) Yes, the dice are fair and independent, so each outcome is equally likely to happen.

(d) There are 9 outcomes where player 1 wins the game and 27 outcomes where player 2 wins. Or stated differently, $n(A) = 9$ and $n(\overline{A}) = 27$. From this we get

$$\mathrm{P}(\overline{A}) = \frac{n(\overline{A})}{n(S)} = \frac{27}{36} = 0.75.$$

Player 2 has a 75% chance of winning.

Question 6

Gamboud reasoned that if the chances of rolling a six is $\frac{1}{6}$, then the chance of seeing a six after 4 throws is

$$4 \cdot \frac{1}{6} = \frac{2}{3}$$

This is not correct, as this would mean that the chance of seeing a six after six games is 100%, which we know to be wrong. However, he was right in that the chance of winning after 4 throws is higher than that of losing. Gamboud used this same logic in the second game, taking the chances of seeing two sixes after a roll as $\frac{1}{36}$ and just multiplied that with 24, which yielded him another $\frac{2}{3}$ chance of winning. This is not the case, as we will prove now.

Let's us correct Gamboud's error and calculate the actual chance of winning both these games. We assume the outcomes to be equiprobable and independent.

(a) There are 6^4 outcomes after four games. The amount of outcomes that don't include a six are 5^4. We first calculate the probability of **not** seeing a six, i.e.

$$\mathrm{P}(A) = \frac{5^4}{6^4} = \frac{625}{1296} = 0.4823 \tag{2}$$

From this we can solve for $\mathrm{P}(\overline{A})$:

$$\mathrm{P}(\overline{A}) = 1 - \mathrm{P}(A) = 0.5177$$

So the chances of winning after four throws of the die are 51.77%.

(b) We use the same method for the second game. There are 35 outcomes in one trial or throw that don't have a double six, so the amount of outcomes with no double six in 24 trials are 35^{24}. The probability of not seeing a double six in 24 dice throws are

$$\mathrm{P}(A) = \frac{35^{24}}{36^{24}} = 0.5086$$

The probability of observing a double six in 24 games are

$$\mathrm{P}(\overline{A}) = 1 - \mathrm{P}(A) = 0.4914$$

(c) Let event A be the event where we don't see a six in N throws.

$$1 - \mathrm{P}(A) = 0.95$$

We want to calculate how many throws will yield a 95 % chance of seeing a six, so we need to use eq. 2:

$$\begin{aligned}\frac{5^N}{6^N} &= 0.05 \\ N &= \log_{\frac{5}{6}}(0.05) \\ N &= 16.43\end{aligned}$$

Question 7

(a) We can reason that if B is a subset of A then B and $A \cap \overline{B}$ are mutually exclusive. This can be written as

$$P(A) = P(B \cup (A \cap \overline{B})).$$

For mutually exclusive events

$$\mathrm{P}(A_1 \cup A_2) = \mathrm{P}(A_1) + \mathrm{P}(A_2),$$

so we can write

$$\mathrm{P}(B \cup (A \cap \overline{B})) = \mathrm{P}(B) + \mathrm{P}(A \cap \overline{B})$$

which leads to

$$\mathrm{P}(A \cap \overline{B}) = \mathrm{P}(A) - \mathrm{P}(B)$$

(b) We know with the axiom of bounded probabilities, $\mathrm{P}(A) \geq 0$, so

$$\mathrm{P}(A \cap \overline{B}) \geq 0$$

and we also know that

$$\mathrm{P}(A \cap \overline{B}) = \mathrm{P}(A) - \mathrm{P}(B)$$

so we can write

$$\mathrm{P}(A) - \mathrm{P}(B) \geq 0$$

which leads to

$$\mathrm{P}(B) \leq \mathrm{P}(A)$$

Question 8

(a)

$$\mathrm{P}(N_G = 2) = \mathrm{P}(\{ggf, gfg, fgg\}) = 0.15 + 0.15 + 0.15 = 0.45$$

(b)

$$\mathrm{P}(N_G \geq 1) = 1 - \mathrm{P}(N_G = 0) = 1 - 0.05 = 0.95$$

(c)

$$\begin{aligned} \mathrm{P}(\{gfg\}|N_G = 2) &= \frac{\mathrm{P}(\{gfg\}, N_G = 2)}{\mathrm{P}(N_G = 2)} \\ &= \frac{\mathrm{P}(\{gfg\})}{\mathrm{P}(N_G = 2)} \\ &= \frac{0.15}{0.45} = \frac{1}{3} \end{aligned}$$

(d)

$$\begin{aligned}
P(N_G = 2|N_G \geq 1) &= \frac{P(N_G = 2, N_G \geq 1)}{P(N_G \geq 1)} \\
&= \frac{P(N_G = 2)}{P(N_G \geq 1)} \\
&= \frac{0.45}{0.95} = \frac{9}{19}
\end{aligned}$$

(e)

$$\begin{aligned}
P(N_G \geq 1|N_G = 2) &= \frac{P(N_G \geq 1, N_G = 2)}{P(N_G = 2)} \\
&= \frac{P(N_G = 2)}{P(N_G = 2)} = 1
\end{aligned}$$

This makes sense, because if there are exactly two good products, then there must be at least one.

(f) To test independence, we check:

$$P(A \cap B) = P(A)P(B)$$

Let $A = \{fff\}$ and $B = \{N_G \geq 1\}$.

$$P(\{fff\} \cap N_G \geq 1) = 0 \quad \text{(since } fff \text{ has } N_G = 0)$$

$$P(\{fff\}) = 0.05 \quad \text{and} \quad P(N_G \geq 1) = 0.95$$

$$P(\{fff\}) \cdot P(N_G \geq 1) = 0.05 \cdot 0.95 = 0.0475 \neq 0$$

Therefore, the events are dependent. Since they are mutually exclusive, knowing one happened implies the other cannot happen.