

Main Questions 1–5; optional bonus Questions 6–7.

All processes are real-valued. Use $R_{XY}(t, \tau) = E[X(t)Y(t + \tau)]$ and frequency f in Hz.

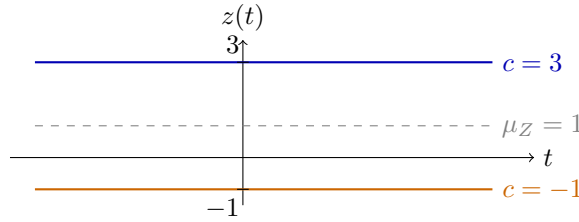
Question 1: Stationarity, ensembles and a constant recording.

The process is $Z(t) = C$, where $P(C = -1) = P(C = 3) = 1/2$. One value of C is drawn per recording and remains fixed throughout it.

(a) Calculate the ensemble properties by averaging over the two possible values of C . All samples within one recording contain the *same* C .

$$\begin{aligned}\mu_Z &= E[C] = (-1)\frac{1}{2} + 3\frac{1}{2} = 1, \\ E[C^2] &= (-1)^2\frac{1}{2} + 3^2\frac{1}{2} = 5, \\ C_Z(\tau) &= \text{Cov}[Z(t), Z(t + \tau)] \\ &= \text{Cov}[C, C] = \text{Var}[C] \\ &= E[C^2] - E[C]^2 = 5 - 1 = 4, \\ R_Z(\tau) &= C_Z(\tau) + \mu_Z^2 = 4 + 1 = 5.\end{aligned}$$

The mean is constant and $R_Z(\tau)$ depends only on the lag, so the process is WSS.



(b) Now consider one recording of the same process: $C = c$, so $z(t) = c$. Its time average uses this fixed level, whereas the ensemble mean in (a) averages over both possible values of C :

$$\frac{1}{2T} \int_{-T}^T z(t) dt = \frac{1}{2T} \int_{-T}^T c dt = c \implies \lim_{T \rightarrow \infty} \bar{z}_T = c \in \{-1, 3\} \neq \mu_Z = 1.$$

Thus the process is **not ergodic in its mean**, even though it is WSS as shown in (a).

(c) For the power spectral density (PSD), use $R_Z(\tau) = 5$ from (a): transform the autocorrelation, not the autocovariance. All its spectral power is at zero frequency.

$$S_Z(f) = \mathcal{F}\{5\} = 5\delta(f), \quad P_Z = R_Z(0) = \int_{-\infty}^{\infty} 5\delta(f) df = 5.$$

Question 2: A biased periodic sensor output.

Lecture review for Question 2(a).

For $V(t) = \cos(t + U)$, the phase U is uniform on $[0, 2\pi)$, with density $1/(2\pi)$. Hence

$$E[V(t)] = \frac{1}{2\pi} \int_0^{2\pi} \cos(t + u) du = \frac{[\sin(t + u)]_0^{2\pi}}{2\pi} = 0.$$

Use $\cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$ with $\alpha = t + u$ and $\beta = t + \tau + u$:

$$\begin{aligned}R_V(t, \tau) &= E[V(t)V(t + \tau)] = \frac{1}{2\pi} \int_0^{2\pi} \cos(t + u) \cos(t + \tau + u) du \\ &= \frac{1}{4\pi} \int_0^{2\pi} [\cos(-\tau) + \cos(2t + \tau + 2u)] du \\ &= \frac{2\pi \cos(-\tau)}{4\pi} + \frac{[\sin(2t + \tau + 2u)]_0^{2\pi}}{8\pi} = \frac{1}{2} \cos \tau.\end{aligned}$$

Here $\cos(-\tau) = \cos \tau$, and the sine endpoints cancel because $\sin(2t + \tau + 4\pi) = \sin(2t + \tau)$.

Question 2 (continued): A biased periodic sensor output.

(a) Write $X(t) = 1 + V(t)$ with $V(t) = \cos(t + U)$. Apply the lecture results reviewed above: $E[V(t)] = 0$ and $R_V(t, \tau) = E[V(t)V(t + \tau)] = \frac{1}{2} \cos \tau$.

$$\begin{aligned}\mu_X(t) &= E[1 + V(t)] = 1, \\ R_X(t, \tau) &= E[(1 + V(t))(1 + V(t + \tau))] \\ &= 1 + E[V(t)] + E[V(t + \tau)] + E[V(t)V(t + \tau)] \\ &= 1 + \frac{1}{2} \cos \tau.\end{aligned}$$

The mean is constant and the autocorrelation depends only on τ , so the process is **WSS**.

(b) A common time shift h replaces U by $U' = (U + h) \bmod 2\pi$. This is still uniform, and the same shifted phase enters every sample:

$$(X(t_1 + h), \dots, X(t_m + h)) = (1 + \cos(t_1 + U'), \dots, 1 + \cos(t_m + U')).$$

Every sampled joint distribution is therefore unchanged: **SSS**.

(c) For a WSS process, the power spectral density is the Fourier transform of its autocorrelation:

$$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau.$$

Substitute $R_X(\tau) = 1 + \frac{1}{2} \cos \tau$ from (a). By linearity,

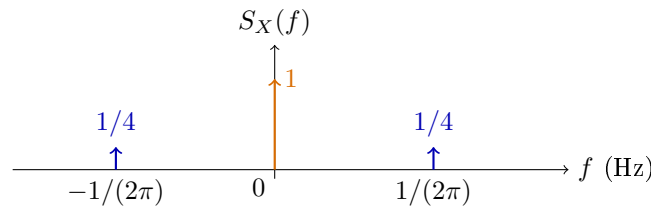
$$S_X(f) = \mathcal{F}\left\{1 + \frac{1}{2} \cos \tau\right\} = \mathcal{F}\{1\} + \frac{1}{2} \mathcal{F}\{\cos \tau\}.$$

The required Fourier pairs, with frequency in Hz, are

$$1 \longleftrightarrow \delta(f), \quad \cos(2\pi f_0 \tau) \longleftrightarrow \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)].$$

Here $2\pi f_0 = 1$, so $f_0 = 1/(2\pi)$. The constant term contributes a DC impulse; the cosine contribution retains both factors of $1/2$:

$$\begin{aligned}S_X(f) &= \delta(f) + \underbrace{\frac{1}{2}}_{\text{factor in } R_X} \underbrace{\frac{1}{2}}_{\text{cosine pair}} \left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right] \\ &= \delta(f) + \frac{1}{4} \left[\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right) \right].\end{aligned}$$



The labels are *impulse weights*, not finite density heights. Each impulse contributes its weight to the spectral integral.

Average power. Calculate it from the zero-lag autocorrelation or the total spectral area:

$$\begin{aligned}P_X &= R_X(0) = 1 + \frac{1}{2} \cos 0 = \frac{3}{2}, \\ P_X &= \int_{-\infty}^{\infty} S_X(f) df = 1 + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}.\end{aligned}$$

The constant bias contributes $P_{DC} = \mu_X^2 = 1$; the fluctuating component contributes $P_{AC} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Question 3: Exponential autocorrelation and spectral width.

(a) The signs of $|\tau|$ differ on the two halves of the integral. Since $a > 0$, both boundary terms at infinity vanish.

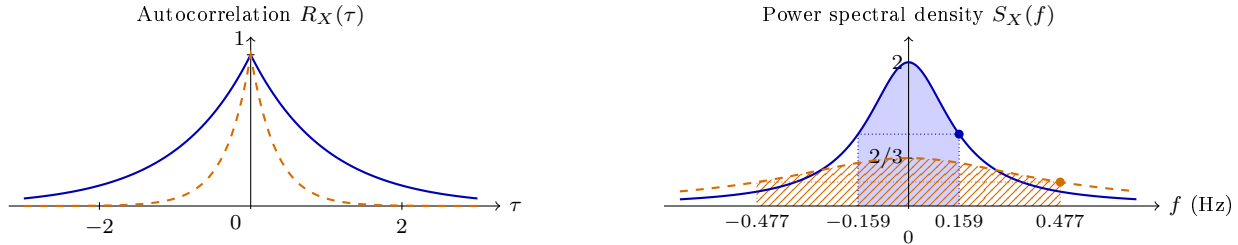
$$\begin{aligned}
 S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau \\
 &= \int_{-\infty}^{\infty} e^{-a|\tau|} e^{-j2\pi f\tau} d\tau \\
 &= \int_{-\infty}^0 e^{(a-j2\pi f)\tau} d\tau + \int_0^{\infty} e^{-(a+j2\pi f)\tau} d\tau \\
 &= \left[\frac{e^{(a-j2\pi f)\tau}}{a-j2\pi f} \right]_{-\infty}^0 + \left[-\frac{e^{-(a+j2\pi f)\tau}}{a+j2\pi f} \right]_0^{\infty} \\
 &= \frac{1}{a-j2\pi f} + \frac{1}{a+j2\pi f} \\
 &= \frac{(a+j2\pi f) + (a-j2\pi f)}{(a-j2\pi f)(a+j2\pi f)} = \frac{2a}{a^2 + 4\pi^2 f^2}.
 \end{aligned}$$

(b) Use $S_X(f)$ from (a). Its peak is $S_X(0) = 2/a$, so the positive half-height frequency satisfies

$$\begin{aligned}
 S_X(f_h) &= \frac{1}{2} S_X(0) \implies \frac{2a}{a^2 + 4\pi^2 f_h^2} = \frac{1}{a} \\
 \implies 2a^2 &= a^2 + 4\pi^2 f_h^2 \implies f_h = \frac{a}{2\pi}.
 \end{aligned}$$

Increasing a makes the correlation decay faster: it becomes **narrower**. The spectrum becomes **broad**er because f_h increases, while its peak $2/a$ decreases.

Illustration: $a = 1$ (solid blue) and $a = 3$ (dashed orange), on the same scales in each panel.



The dots mark each curve's own half-height: $f_h = 1/(2\pi) \approx 0.159$ Hz for $a = 1$, and $f_h = 3/(2\pi) \approx 0.477$ Hz for $a = 3$. Blue fill and orange hatching show each curve's band $[-f_h, f_h]$: each region has area $1/2$, whereas each complete spectrum has area 1 , as calculated in (c).

(c) The total average power is the zero-lag autocorrelation:

$$P_X = R_X(0) = e^0 = 1.$$

The power retained in the measurement band is the shaded/hatched area under its PSD between $-f_h$ and f_h . Substitute $f_h = a/(2\pi)$ from (b).

To integrate, set $z = 2\pi f/a$, so $df = a dz/(2\pi)$ and $a^2 + 4\pi^2 f^2 = a^2(1 + z^2)$. The frequency limits $f = \pm a/(2\pi)$ become $z = \pm 1$. We can then use $\int (1 + z^2)^{-1} dz = \arctan z$:

$$\begin{aligned}
 P_{\text{band}} &= \int_{-f_h}^{f_h} S_X(f) df = \int_{-a/(2\pi)}^{a/(2\pi)} \frac{2a}{a^2 + 4\pi^2 f^2} df \\
 &= \int_{-1}^1 \frac{2a}{a^2(1 + z^2)} \frac{a}{2\pi} dz \\
 &= \frac{1}{\pi} \int_{-1}^1 \frac{1}{1 + z^2} dz = \frac{1}{\pi} [\arctan z]_{-1}^1 \\
 &= \frac{1}{\pi} [\arctan(1) - \arctan(-1)] = \frac{1}{\pi} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) = \frac{1}{2}.
 \end{aligned}$$

Thus the measurement retains $P_{\text{band}}/P_X = (1/2)/1 = 1/2$, or 50%, of the total average power. This follows from the integral, not merely from the half-height definition.

Question 3 (continued): Exponential autocorrelation and spectral width.

(d) The Dirac impulse transforms to a constant. Thus ideal continuous-time white noise has flat PSD over all frequencies:

$$S_W(f) = \mathcal{F}\{\eta_0 \delta(\tau)\} = \eta_0, \quad P_W = \int_{-\infty}^{\infty} \eta_0 df = \infty.$$

It is an idealised model, not a physically observable signal of finite power. Restricting the noise to a finite bandwidth produces finite observed power; for example, a unit-gain passband $|f| \leq B$ contains power $2B\eta_0$.

Question 4: Radar delay and ordered cross-correlation.

(a) Use linearity for the mean and expand the product for autocorrelation. Independence and zero means make both signal/noise cross terms zero.

$$\begin{aligned} \mu_Y &= E[X(t-d)] + E[N(t)] = 0, \\ R_Y(t, \tau) &= E[(X(t-d) + N(t))(X(t+\tau-d) + N(t+\tau))] \\ &= R_X(\tau) + E[X(t-d)]E[N(t+\tau)] + E[N(t)]E[X(t+\tau-d)] + R_N(\tau) \\ &= e^{-a|\tau|} + \beta e^{-b|\tau|}. \end{aligned}$$

A constant mean and lag-only autocorrelation show that Y is WSS. Delaying both signal samples does not change their separation.

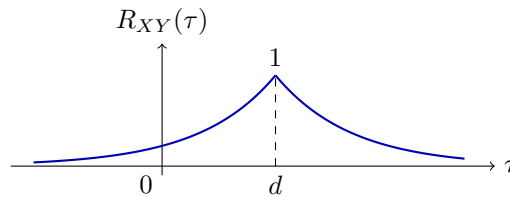
(b) The order matters: the *second* process is evaluated at $t + \tau$. Expand each product before simplifying. Since the processes X and N are independent, their samples are independent at any pair of times. Each mixed expectation therefore factors into the product of their zero means:

$$\begin{aligned} R_{XY}(t, \tau) &= E[X(t)(X(t+\tau-d) + N(t+\tau))] \\ &= E[X(t)X(t+\tau-d)] + E[X(t)N(t+\tau)] \\ &= R_X(\tau-d) + \underbrace{E[X(t)]E[N(t+\tau)]}_{0 \cdot 0 = 0} \\ &= e^{-a|\tau-d|}, \\ R_{YX}(t, \tau) &= E[(X(t-d) + N(t))X(t+\tau)] \\ &= E[X(t-d)X(t+\tau)] + E[N(t)X(t+\tau)] \\ &= R_X(\tau+d) + \underbrace{E[N(t)]E[X(t+\tau)]}_{0 \cdot 0 = 0} \\ &= e^{-a|\tau+d|}, \\ R_{YX}(-\tau) &= e^{-a|-\tau+d|} = e^{-a|\tau-d|} = R_{XY}(\tau). \end{aligned}$$

The signal-only terms use the sample-time differences $(t + \tau - d) - t = \tau - d$ and $(t + \tau) - (t - d) = \tau + d$, respectively.

Both processes are WSS and their cross-correlation depends only on lag: they are **jointly WSS**.

(c) The cross-correlation peaks where $|\tau - d| = 0$: at the positive round-trip delay $\tau = d$, with peak value one. The schematic shows this displaced peak.



$$2r = cd \implies r = \frac{cd}{2} = \frac{(3 \times 10^8)(4 \times 10^{-6})}{2} = 600 \text{ m.}$$

Question 5: A correlated Gaussian sequence from independent input samples.

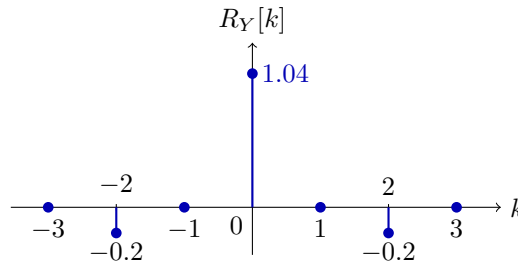
(a) Input samples have mean zero and variance one. Thus $E[X_i^2] = 1$, whereas $E[X_i X_j] = E[X_i] E[X_j] = 0$ for $i \neq j$. Equivalently, $E[X_i X_j] = \delta[j - i]$.

$$\begin{aligned} E[Y_n] &= E[X_n] - 0.2 E[X_{n-2}] = 0, \\ R_Y[n, k] &= E[(X_n - 0.2X_{n-2})(X_{n+k} - 0.2X_{n+k-2})] \\ &= E[X_n X_{n+k}] - 0.2 E[X_n X_{n+k-2}] - 0.2 E[X_{n-2} X_{n+k}] + 0.04 E[X_{n-2} X_{n+k-2}] \\ &= \delta[k] - 0.2\delta[k-2] - 0.2\delta[k+2] + 0.04\delta[k] \\ &= 1.04\delta[k] - 0.2\delta[k-2] - 0.2\delta[k+2]. \end{aligned}$$

At $k = 0$, the first and fourth products pair identical indices. At $k = 2$, only the second product does; at $k = -2$, only the third does. All other lags pair distinct input samples throughout.

$$R_Y[k] = \begin{cases} 1.04, & k = 0, \\ -0.2, & k = \pm 2, \\ 0, & \text{all other integer } k. \end{cases}$$

(b) The discrete sketch includes a stem at zero, two negative stems and zero-valued samples at the other displayed lags. There are no continuous-time impulses.



Equal values at opposite lags confirm symmetry about $k = 0$:

$$P_Y = E[Y_n^2] = R_Y[0] = 1.04, \quad R_Y[-k] = R_Y[k], \quad |R_Y[k]| \in \{0, 0.2, 1.04\} \leq 1.04.$$

(c) Every finite vector of output samples is a linear transformation of a finite vector of independent Gaussian inputs. It is therefore jointly Gaussian: Y_n is a **Gaussian sequence**.

Use the result already derived in (a):

$$R_Y[n, k] = E[Y_n Y_{n+k}] = 1.04\delta[k] - 0.2\delta[k-2] - 0.2\delta[k+2].$$

Only the lag k appears, not n , so write this as $R_Y[k]$. The mean $E[Y_n] = 0$ is constant: the sequence is **WSS**. To relate autocorrelation to covariance, start from the covariance definition and substitute the zero means:

$$\begin{aligned} \text{Cov}[Y_n, Y_{n+k}] &= E[Y_n Y_{n+k}] - E[Y_n] E[Y_{n+k}] \\ &= R_Y[k] - 0 \cdot 0 \\ &= R_Y[k] = C_Y[k]. \end{aligned}$$

The means and covariances are therefore unchanged by a common index shift. Since the sequence is jointly Gaussian, these determine every finite joint distribution, so it is also **SSS**.

For the two pairs in the question, use lags $k = 1$ and $k = 2$. Recall that $\delta[0] = 1$ and $\delta[k] = 0$ for nonzero integer k :

$$\begin{aligned} \text{Cov}[Y_n, Y_{n+1}] &= R_Y[1] = 1.04\delta[1] - 0.2\delta[-1] - 0.2\delta[3] \\ &= 1.04(0) - 0.2(0) - 0.2(0) = 0 \implies \text{independent}, \\ \text{Cov}[Y_n, Y_{n+2}] &= R_Y[2] = 1.04\delta[2] - 0.2\delta[0] - 0.2\delta[4] \\ &= 1.04(0) - 0.2(1) - 0.2(0) = -0.2 \neq 0 \implies \text{dependent}. \end{aligned}$$

The first independence conclusion uses *joint Gaussianity*, not zero covariance alone. Adjacent outputs also use disjoint sets of independent inputs. The sequence is **not white**, since $C_Y[2] = -0.2 \neq 0$ at a nonzero lag (and similarly at $k = -2$).

Optional bonus solutions**Question 6: Shifting, scaling and reversing time.**

For a WSS process, $E[X(s)X(u)] = R_X(u - s)$. Use the actual separation between the two input times.

(a) A common time shift does not change the mean or the separation within the transformed process.

$$\begin{aligned}\mu_{X_1} &= E[X(t+h)] = \mu_X, \\ R_{X_1}(t, \tau) &= E[X(t+h)X(t+\tau+h)] = R_X(\tau), \\ R_{XX_1}(t, \tau) &= E[X(t)X(t+\tau+h)] = R_X(\tau+h).\end{aligned}$$

X_1 is WSS, and the cross-correlation depends only on τ : X and X_1 are **jointly WSS**.

(b) Scaling changes the separation within X_2 to $a\tau$. In the cross-correlation, only the second process's time is scaled.

$$\begin{aligned}\mu_{X_2} &= E[X(at)] = \mu_X, \\ R_{X_2}(t, \tau) &= E[X(at)X(at+a\tau)] = R_X(a\tau), \\ R_{XX_2}(t, \tau) &= E[X(t)X(a(t+\tau))] = R_X((a-1)t+a\tau).\end{aligned}$$

X_2 is always WSS. When $a = 1$, the pair is jointly WSS. When $a \neq 1$, the argument sweeps all real values as t varies, so joint WSS holds only if R_X is constant. It is *not guaranteed* for a general WSS input.

(c) Put $a = -1$. A real-valued WSS autocorrelation is even.

$$\mu_{X_3} = \mu_X, \quad R_{X_3}(\tau) = R_X(-\tau) = R_X(\tau), \quad R_{XX_3}(t, \tau) = R_X(-2t - \tau).$$

Time reversal therefore preserves WSS of the individual process, but it does not generally preserve joint WSS with the original process. The constant- R_X exception still applies.

Question 7: Sum and difference of independent processes.

Independence and zero means give $R_{XY}(\tau) = R_{YX}(\tau) = 0$. Both output means are zero.

(a) Proceed as in Question 4(a): expand the two output products and use independence and zero means to eliminate the cross terms. In the difference's autocorrelation, the product of its two minus signs gives a plus.

$$\begin{aligned}R_{W_1}(\tau) &= R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau) \\ &= e^{-|\tau|} + \cos(2\pi\tau), \\ R_{W_2}(\tau) &= R_X(\tau) - R_{XY}(\tau) - R_{YX}(\tau) + R_Y(\tau) \\ &= e^{-|\tau|} + \cos(2\pi\tau), \\ P_{W_1} &= R_{W_1}(0) = 1 + 1 = 2, \quad P_{W_2} = R_{W_2}(0) = 2.\end{aligned}$$

(b) Only the *second* output has a minus sign in this cross product.

$$\begin{aligned}R_{W_1W_2}(t, \tau) &= E[(X(t) + Y(t))(X(t+\tau) - Y(t+\tau))] \\ &= R_X(\tau) - R_{XY}(\tau) + R_{YX}(\tau) - R_Y(\tau) \\ &= e^{-|\tau|} - \cos(2\pi\tau).\end{aligned}$$

Both outputs are WSS and this cross-correlation depends only on lag. They are therefore **jointly WSS**.

(c) Apply Fourier linearity to the difference of autocorrelations.

$$S_{W_1W_2}(f) = \frac{2}{1 + 4\pi^2 f^2} - \frac{1}{2}[\delta(f-1) + \delta(f+1)].$$

A cross spectrum describes the relationship between two *different* processes. It is not the nonnegative power density of one process, so the negative impulse weights are valid. Nonnegativity applies to S_{W_1} and S_{W_2} , not to $S_{W_1W_2}$.