

Topics covered in this tutorial

- Strict-sense and wide-sense stationarity (SSS and WSS)
- Ensemble/time averages, mean ergodicity and average power
- Autocorrelation, ordered cross-correlation and delay estimation
- Gaussian processes, white noise and correlated Gaussian sequences
- Power spectral density (PSD) and cross-spectral density

Complete all main questions (1–5) before the tutorial test that follows. Bonus Questions 6 and 7 are optional. The test normally starts between approximately 16:00 and 16:15.

All processes are real-valued. Use the lecture conventions below, with frequency f in Hz. Continuous-time $\delta(\tau)$ is a Dirac impulse; discrete-time $\delta[k]$ is the Kronecker delta.

$$R_X(t, \tau) = E[X(t)X(t + \tau)], \quad R_{XY}(t, \tau) = E[X(t)Y(t + \tau)], \quad S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau.$$

Onderwerpe wat in hierdie tutoriaal gedek word

- Strengsin- en wydsin-stasionariteit (SSS en WSS)
- Ensemble-/tydgemiddeldes, gemiddelde-ergodisiteit en gemiddelde drywing
- Outokorrelasie, geordende kruiskorrelasie en verdragingskatting
- Gaussiese prosesse, witruis en gekorreleerde Gaussiese rye
- Drywingsdigtheidspektrum (DDS) en kruisspektrale digtheid

Voltooi al die hoofvrae (1–5) voor die tutoriaaltoets wat daarop volg. Bonusvrae 6 en 7 is opsioneel. Die toets begin gewoonlik tussen ongeveer 16:00 en 16:15.

Alle prosesse is reëelwaardig. Gebruik die lesings se onderstaande konvensies, met frekwensie f in Hz. Kontinue-tyd- $\delta(\tau)$ is 'n Dirac-impuls; diskrete-tyd- $\delta[k]$ is die Kronecker-delta.

1. A sensor's calibration level is $Z(t) = C$ for every $t \in \mathbb{R}$, where $P(C = -1) = P(C = 3) = 1/2$. One value of C is drawn per recording and stays fixed throughout it.
 - (a) Find the ensemble mean μ_Z , autocovariance $C_Z(\tau)$ and autocorrelation $R_Z(\tau)$. Explain whether this process is WSS.
 - (b) For the same process, one recording has $C = c$ and hence $z(t) = c$. Calculate $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T z(t) dt$ and compare it with the ensemble mean in (a). Is the process ergodic in its mean?
 - (c) Using $R_Z(\tau)$ from (a), find the power spectral density (PSD) of $Z(t)$ and its average power. Use the Fourier pair $1 \longleftrightarrow \delta(f)$.

2. A sensor output with a constant bias is modelled by $X(t) = 1 + \cos(t + U)$, $t \in \mathbb{R}$, where U is uniform on $[0, 2\pi)$.

(a) Using the mean and autocorrelation of the zero-mean random-phase sinusoid derived in class, find μ_X and $R_X(\tau)$, and show that $X(t)$ is WSS.

(b) Explain why a common shift of all observation times leaves every joint distribution unchanged. Is this process SSS?

(c) Find and sketch $S_X(f)$, labelling the impulse locations and weights. Calculate the average power from both $R_X(0)$ and the spectrum, identifying the contributions from the constant bias and the fluctuating component.

1. 'n Sensor se kalibrasievlak is $Z(t) = C$ vir elke $t \in \mathbb{R}$, waar $P(C = -1) = P(C = 3) = 1/2$. Een waarde van C word per opname getrek en bly deurgaans vas.

(a) Vind die ensembledigtheid μ_Z , outokovariansie $C_Z(\tau)$ en outokorrelasie $R_Z(\tau)$. Verduidelik of hierdie proses WSS is.

(b) Vir dieselfde proses het een opname $C = c$ en dus $z(t) = c$. Bereken $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T z(t) dt$ en vergelyk dit met die ensembledigtheid in (a). Is die proses ergodies in sy gemiddelde?

(c) Gebruik $R_Z(\tau)$ uit (a) om die drywingsdigtheidspektrum (DDS) van $Z(t)$ en sy gemiddelde drywing te vind. Gebruik die Fourier-paar $1 \longleftrightarrow \delta(f)$.

2. 'n Sensoruitset met 'n konstante voorspanning word gemodelleer deur $X(t) = 1 + \cos(t + U)$, $t \in \mathbb{R}$, waar U uniform op $[0, 2\pi)$ is.

(a) Gebruik die gemiddelde en outokorrelasie van die nulgemiddelde sinusvormige sein met ewekansige fase wat in die klas afgelei is om μ_X en $R_X(\tau)$ te vind, en toon dat $X(t)$ WSS is.

(b) Verduidelik waarom 'n gemeenskaplike verskuiwing van alle waarnemingstye elke gesamentlike verdeling onveranderd laat. Is hierdie proses SSS?

(c) Vind en skets $S_X(f)$, en dui die impulsposisies en -gewigte aan. Bereken die gemiddelde drywing uit sowel $R_X(0)$ as die spektrum, en identifiseer die bydraes van die konstante voorspanning en die wisselende komponent.

3. A zero-mean WSS process has $R_X(\tau) = e^{-a|\tau|}$, where $a > 0$.

(a) Derive $S_X(f)$ from the Fourier integral. Split the integral at $\tau = 0$ and show how the two terms combine.

(b) Using $S_X(f)$ from (a), determine the positive frequency f_h for which $S_X(f_h) = S_X(0)/2$. If a increases, does the autocorrelation become narrower or broader, and what happens to the spectrum?

(c) Calculate the total average power and the fraction of this power retained by a measurement that passes frequencies $|f| \leq f_h$ unchanged and rejects all others. Hint: $\int (1 + u^2)^{-1} du = \arctan u$.

(d) Ideal continuous-time white Gaussian noise has $R_W(\tau) = \eta_0 \delta(\tau)$, with $\eta_0 > 0$. Give its PSD. Does this ideal model have finite total average power? Explain why a finite measurement bandwidth is important.

4. A received radar signal is $Y(t) = X(t - d) + N(t)$, where $d > 0$ is the round-trip delay. The zero-mean WSS processes X and N are independent, with $R_X(\tau) = e^{-a|\tau|}$ and $R_N(\tau) = \beta e^{-b|\tau|}$, where $a, b, \beta > 0$.

(a) Derive μ_Y and $R_Y(\tau)$. Explain why the signal/noise cross terms vanish and show that Y is WSS.

(b) Find $R_{XY}(\tau)$ and $R_{YX}(\tau)$ using the stated order of the processes. Are X and Y jointly WSS? Verify $R_{XY}(\tau) = R_{YX}(-\tau)$.

(c) Sketch $R_{XY}(\tau)$ and show how its peak identifies d . A measured peak occurs at $4 \mu\text{s}$. Calculate the target distance, using propagation speed $c = 3 \times 10^8$ m/s and round-trip distance $2r = cd$.

5. Let $X_n, n \in \mathbb{Z}$, be iid Gaussian random variables with $\mu_X = 0$ and $\sigma_X = 1$. Define $Y_n = X_n - 0.2X_{n-2}$.

(a) Find $E[Y_n]$ and derive $R_Y[n, k] = E[Y_n Y_{n+k}]$. Expand all four products and identify the lags at which input indices coincide. Express the final answer using Kronecker deltas.

(b) Sketch $R_Y[k]$ for $k = -3, \dots, 3$ and calculate the average power. Check the symmetry about $k = 0$ and the bound $|R_Y[k]| \leq R_Y[0]$.

(c) Is Y_n a Gaussian sequence, WSS and SSS? Is it white? Explain whether Y_n and Y_{n+1} are independent, and whether Y_n and Y_{n+2} are independent.

3. 'n Nulgemiddelde WSS-proses het $R_X(\tau) = e^{-a|\tau|}$, waar $a > 0$.

(a) Lei $S_X(f)$ uit die Fourier-integraal af. Verdeel die integraal by $\tau = 0$ en toon hoe die twee terme saamgevoeg word.

(b) Gebruik $S_X(f)$ uit (a) om die positiewe frekwensie f_h te bepaal waarvoor $S_X(f_h) = S_X(0)/2$. Indien a toeneem, word die outokorrelasie nouer of breëer, en wat gebeur met die spektrum?

(c) Bereken die totale gemiddelde drywing en die breuk van hierdie drywing wat behou word deur 'n meting wat frekwensies $|f| \leq f_h$ onveranderd deurlaat en alle ander verwerp. Wenk: $\int (1 + u^2)^{-1} du = \arctan u$.

(d) Ideale wit Gaussiese ruis in kontinue tyd het $R_W(\tau) = \eta_0 \delta(\tau)$, met $\eta_0 > 0$. Gee sy DDS. Het hierdie ideale model eindige totale gemiddelde drywing? Verduidelik waarom 'n eindige meetbandwydte belangrik is.

4. 'n Ontvang radarsein is $Y(t) = X(t - d) + N(t)$, waar $d > 0$ die heen-en-terug-vertraging is. Die nulgemiddelde WSS-prosesse X en N is onafhanklik, met $R_X(\tau) = e^{-a|\tau|}$ en $R_N(\tau) = \beta e^{-b|\tau|}$, waar $a, b, \beta > 0$.

(a) Lei μ_Y en $R_Y(\tau)$ af. Verduidelik waarom die sein-/ruis-kruisterme verdwyn en toon dat Y WSS is.

(b) Vind $R_{XY}(\tau)$ en $R_{YX}(\tau)$ deur die vermelde volgorde van die prosesse te gebruik. Is X en Y gesamentlik WSS? Verifieer $R_{XY}(\tau) = R_{YX}(-\tau)$.

(c) Skets $R_{XY}(\tau)$ en toon hoe sy maksimum d identifiseer. 'n Gemete maksimum kom by $4 \mu\text{s}$ voor. Bereken die teikenafstand, met voortplantingspoed $c = 3 \times 10^8$ m/s en heen-en-terug-afstand $2r = cd$.

5. Laat $X_n, n \in \mathbb{Z}$, oiv Gaussiese toevalsveranderlikes met $\mu_X = 0$ en $\sigma_X = 1$ wees. Definieer $Y_n = X_n - 0.2X_{n-2}$.

(a) Vind $E[Y_n]$ en lei $R_Y[n, k] = E[Y_n Y_{n+k}]$ af. Brei al vier produkte uit en identifiseer die lagwaardes waarby insetindekse saamval. Druk die finale antwoord met Kronecker-deltas uit.

(b) Skets $R_Y[k]$ vir $k = -3, \dots, 3$ en bereken die gemiddelde drywing. Kontroleer die simmetrie om $k = 0$ en die grens $|R_Y[k]| \leq R_Y[0]$.

(c) Is Y_n 'n Gaussiese ry, WSS en SSS? Is dit wit? Verduidelik of Y_n en Y_{n+1} onafhanklik is, en of Y_n en Y_{n+2} onafhanklik is.

Optional bonus practice

6. Let $X(t)$ be a real-valued WSS process on $\mathbb{R}$ with mean μ_X and autocorrelation $R_X(\tau)$. For each transformation, determine its mean and autocorrelation and whether it is necessarily jointly WSS with X .

(a) $X_1(t) = X(t + h)$, where $h \in \mathbb{R}$. Derive $R_{X X_1}(t, \tau)$.

(b) $X_2(t) = X(at)$, where $a \in \mathbb{R}$. Derive $R_{X X_2}(t, \tau)$ and discuss why joint WSS is not guaranteed when $a \neq 1$. Include the special case of constant R_X .

(c) Use (b) with $a = -1$ for time reversal $X_3(t) = X(-t)$. Why is $R_{X_3}(\tau) = R_X(\tau)$ even though joint WSS with X is not guaranteed?

7. The independent zero-mean WSS processes $X(t)$ and $Y(t)$ have $R_X(\tau) = e^{-|\tau|}$ and $R_Y(\tau) = \cos(2\pi\tau)$. Define $W_1(t) = X(t) + Y(t)$ and $W_2(t) = X(t) - Y(t)$.

(a) Find $R_{W_1}(\tau)$, $R_{W_2}(\tau)$ and their average powers.

(b) Find $R_{W_1 W_2}(\tau)$ and explain whether W_1 and W_2 are jointly WSS.

(c) Give $S_{W_1 W_2}(f)$. Explain why negative terms here do not violate the nonnegativity requirement for an *auto* PSD.

Opsionele bonusoefeninge

6. Laat $X(t)$ 'n reëelwaardige WSS-proses op $\mathbb{R}$ wees met gemiddelde μ_X en outokorrelasie $R_X(\tau)$. Bepaal vir elke transformasie sy gemiddelde en outokorrelasie en of dit noodwendig gesamentlik WSS met X is.

(a) $X_1(t) = X(t + h)$, waar $h \in \mathbb{R}$. Lei $R_{X X_1}(t, \tau)$ af.

(b) $X_2(t) = X(at)$, waar $a \in \mathbb{R}$. Lei $R_{X X_2}(t, \tau)$ af en bespreek waarom gesamentlike WSS nie gewaarborg word wanneer $a \neq 1$ nie. Sluit die spesiale geval van konstante R_X in.

(c) Gebruik (b) met $a = -1$ vir tydomkering $X_3(t) = X(-t)$. Waarom is $R_{X_3}(\tau) = R_X(\tau)$ hoewel gesamentlike WSS met X nie gewaarborg word nie?

7. Die onafhanklike nulgemiddelde WSS-prosesse $X(t)$ en $Y(t)$ het $R_X(\tau) = e^{-|\tau|}$ en $R_Y(\tau) = \cos(2\pi\tau)$. Definieer $W_1(t) = X(t) + Y(t)$ en $W_2(t) = X(t) - Y(t)$.

(a) Vind $R_{W_1}(\tau)$, $R_{W_2}(\tau)$ en hul gemiddelde drywings.

(b) Vind $R_{W_1 W_2}(\tau)$ en verduidelik of W_1 en W_2 gesamentlik WSS is.

(c) Gee $S_{W_1 W_2}(f)$. Verduidelik waarom negatiewe terme hier nie die nienegatiwiteitsvereiste vir 'n *auto*-DDS oortree nie.