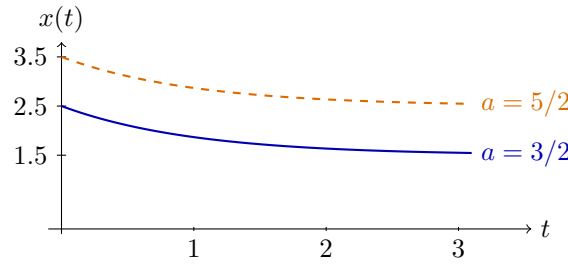


Questions 1–5. The lag convention is $C_X(t, \tau) = \text{Cov}[X(t), X(t + \tau)]$. Gaussian parameters are stated as mean and standard deviation.

Question 1: A shared random offset: sample functions and process averages.

(a) This is a continuous-time, continuous-valued process. Fixing $A = a$ gives the deterministic curve $x(t) = e^{-t} + a$. At a fixed time, A remains random, so $X(t)$ is a random variable.



(b) For fixed t , adding e^{-t} shifts the uniform density; its width remains $3 - 1 = 2$ and its height is $1/2$.

$$f_{X(t)}(x) = \begin{cases} \frac{1}{2}, & e^{-t} + 1 \leq x \leq e^{-t} + 3, \\ 0, & \text{otherwise.} \end{cases} \quad E[A] = 2, \quad E[A^2] = \frac{13}{3}.$$

$$\mu_X(t) = E[e^{-t} + A] = e^{-t} + 2,$$

$$E[X(t)^2] = E[e^{-2t} + 2e^{-t}A + A^2] = e^{-2t} + 4e^{-t} + \frac{13}{3},$$

$$\begin{aligned} \text{Var}[X(t)] &= E[X(t)^2] - \mu_X(t)^2 \\ &= \left(e^{-2t} + 4e^{-t} + \frac{13}{3} \right) - (e^{-2t} + 4e^{-t} + 4) = \frac{1}{3}. \end{aligned}$$

(c) Subtracting the ensemble means cancels the deterministic offsets: $X(t) - \mu_X(t) = A - 2$ and $X(t + \tau) - \mu_X(t + \tau) = A - 2$. Both samples contain the *same* A , so

$$\begin{aligned} C_X(t, \tau) &= \text{Cov}[X(t), X(t + \tau)] \\ &= E[(X(t) - \mu_X(t))(X(t + \tau) - \mu_X(t + \tau))] \\ &= E[(A - 2)^2] = E[(A - E[A])^2] = \text{Var}[A] = \frac{(3 - 1)^2}{12} = \frac{1}{3}, \\ R_X(t, \tau) &= C_X(t, \tau) + \mu_X(t)\mu_X(t + \tau) \\ &= \frac{1}{3} + (e^{-t} + 2)(e^{-(t+\tau)} + 2) \\ &= e^{-(2t+\tau)} + 2e^{-t} + 2e^{-(t+\tau)} + \frac{13}{3} \quad (\text{optional expansion}). \end{aligned}$$

The samples are dependent: observing one reveals A and hence every other sample. Their nonzero covariance also rules out independence.

(d) A time average integrates one realised curve, rather than averaging different outcomes at a fixed time. For $T > 0$:

$$\begin{aligned} \bar{x}_T &= \frac{1}{T} \int_0^T (e^{-t} + a) dt \\ &= \frac{1}{T} [-e^{-t} + at]_0^T \\ &= \frac{1 - e^{-T}}{T} + a \rightarrow a \quad (T \rightarrow \infty). \end{aligned}$$

The limit depends on this sample's $a \in [1, 3]$, whereas the ensemble mean is $e^{-t} + 2$ and tends to 2. A long record does not average over the possible offsets.

Question 2: Service-centre arrivals: counts, increments and waiting time.

The arrival rate is $\lambda = 1/5 = 0.2$ per minute. Successive interarrival times Y_i are iid exponential with mean $1/\lambda = 5$ minutes.

(a) Service starts at the fourth arrival, $T_4 = \sum_{i=1}^4 Y_i$. By linearity of expectation:

$$E[T_4] = \sum_{i=1}^4 E[Y_i] = \frac{4}{\lambda} = 20 \text{ min} \Rightarrow 09:20.$$

This is an expected time, not a guaranteed start time.

(b) Because counts only increase, service has not started by 60 minutes exactly when $N(60) \leq 3$.

$$\begin{aligned} P(N(60) \leq 3) &= \sum_{n=0}^3 e^{-\lambda 60} \frac{(\lambda 60)^n}{n!} \\ &= e^{-12} \left(1 + 12 + \frac{12^2}{2!} + \frac{12^3}{3!} \right) = 373e^{-12} \approx 0.002292. \end{aligned}$$

(c) The increments $N(10)$ and $N(20) - N(10)$ count arrivals in $(0, 10]$ and $(10, 20]$. They are independent, each with Poisson parameter $0.2(10) = 2$. For integers $0 \leq k \leq \ell$, the joint PMF is

$$\begin{aligned} p_{N(10), N(20)}(k, \ell) &= P(N(10) = k, N(20) = \ell) \\ &= P(N(10) = k) P(N(20) - N(10) = \ell - k) \\ &= \left(e^{-2} \frac{2^k}{k!} \right) \left(e^{-2} \frac{2^{\ell-k}}{(\ell-k)!} \right) = e^{-4} \frac{2^\ell}{k!(\ell-k)!}. \end{aligned}$$

The joint PMF is zero otherwise. Substituting $k = 1$ and $\ell = 3$ gives

$$\begin{aligned} P(N(10) = 1, N(20) = 3) &= P(N(10) = 1) P(N(20) - N(10) = 2) \\ &= \left(e^{-2} \frac{2^1}{1!} \right) \left(e^{-2} \frac{2^2}{2!} \right) = 4e^{-4} \approx 0.073263. \end{aligned}$$

$N(20)$ includes $N(10)$, so the cumulative counts are dependent. For example, $N(10) = 1$ makes $N(20) = 0$ impossible.

(d) The exponential survival probability is $P(T_1 > s) = e^{-\lambda s}$. Hence

$$P(T_1 > 10 \mid T_1 > 4) = \frac{P(T_1 > 10)}{P(T_1 > 4)} = \frac{e^{-0.2(10)}}{e^{-0.2(4)}} = e^{-0.2(6)} = e^{-1.2} \approx 0.301194.$$

By memorylessness, this is also $P(T_1 > 6) = e^{-0.2(6)}$. After waiting four minutes, the probability of waiting another six minutes is the same as the original probability of waiting more than six minutes.

Question 3: Combining and thinning independent Poisson notification streams.

(a) Superposition of independent Poisson processes adds their rates.

$$\lambda = \lambda_A + \lambda_B = 2 + 1 = 3 \text{ min}^{-1}, \quad P(N(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

$$P(N(2) = 4) = e^{-6} \frac{6^4}{4!} = 54e^{-6} \approx 0.133853.$$

(b) Independent Bernoulli labelling thins the merged process into *independent* priority and routine Poisson processes.

$$\lambda_P = p\lambda = \frac{1}{4}(3) = 0.75 \text{ min}^{-1}, \quad \lambda_R = (1-p)\lambda = \frac{3}{4}(3) = 2.25 \text{ min}^{-1}.$$

$$\begin{aligned} P(N_P(2) = 1, N_R(2) = 3) &= P(N_P(2) = 1)P(N_R(2) = 3) \\ &= \left(e^{-1.5} \frac{1.5^1}{1!} \right) \left(e^{-4.5} \frac{4.5^3}{3!} \right) \\ &= 22.78125e^{-6} \approx 0.056469. \end{aligned}$$

This is an unconditional independence result. If the total number were fixed, the two split counts would have to add to that total; that is a different conditioning question.

(c) In a merged Poisson process, the next arrival's source probability is proportional to its rate.

$$P(\text{next from A}) = \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{2}{2+1} = \frac{2}{3}.$$

Question 4: Brownian temperature changes: increments and joint samples.

(a) For Brownian motion, an increment of duration h has mean zero and variance αh .

$$\text{Var}[W(12) - W(0)] = 12\alpha = \left(\frac{3}{5}\right)^2 \Rightarrow \alpha = \frac{0.36}{12} = 0.03 \frac{^\circ\text{C}^2}{\text{h}}.$$

$$\Delta = W(5) - W(2) : \quad \text{Gaussian}(\mu_\Delta = 0, \sigma_\Delta = \sqrt{\alpha(5-2)} = 0.3).$$

(b) Put $U = W(2)$ and $\Delta = W(5) - W(2)$. Independent increments make U and Δ independent, with variances 2α and 3α . Given $W(2) = w_2$, we have $W(5) = w_2 + \Delta$: its conditional PDF is the increment PDF shifted by w_2 , without changing its height. Use joint PDF = marginal PDF $\times$ conditional PDF.

$$\begin{aligned} f_{W(2), W(5)}(w_2, w_5) &= f_{W(2)}(w_2) f_{W(5)|W(2)}(w_5 | w_2) \\ &= f_U(w_2) f_\Delta(w_5 - w_2) \\ &= \frac{e^{-w_2^2/(4\alpha)}}{\sqrt{2\pi(2\alpha)}} \frac{e^{-(w_5 - w_2)^2/(6\alpha)}}{\sqrt{2\pi(3\alpha)}} \\ &= \frac{1}{2\pi(0.03)\sqrt{6}} \exp\left[-\frac{w_2^2}{0.12} - \frac{(w_5 - w_2)^2}{0.18}\right], \quad w_2, w_5 \in \mathbb{R}. \end{aligned}$$

The second factor is an *increment* density, not the marginal density of $W(5)$.

(c) First suppose $0 \leq s \leq u$, and write $W(u) = W(s) + [W(u) - W(s)]$. Split covariance over this sum:

$$\begin{aligned} \text{Cov}[W(s), W(u)] &= \text{Cov}[W(s), W(s) + [W(u) - W(s)]] \\ &= \text{Cov}[W(s), W(s)] + \text{Cov}[W(s), W(u) - W(s)] \\ &= \text{Var}[W(s)] + 0 = \alpha s. \end{aligned}$$

The mixed covariance is zero because $W(s) - W(0)$ and $W(u) - W(s)$ are increments over nonoverlapping intervals, hence independent. Also, $\text{Cov}[W(s), W(s)] = \text{Var}[W(s)]$. If $u < s$, interchange the times to obtain αu . Therefore, in general:

$$\text{Cov}[W(s), W(u)] = \alpha \min(s, u), \quad s, u \geq 0.$$

Since $\mu_W(t) = 0$, autocovariance and autocorrelation are equal:

$$\begin{aligned} C_W(t, \tau) &= \alpha \min(t, t + \tau), \quad t, t + \tau \geq 0, \\ R_W(t, \tau) &= C_W(t, \tau) + \mu_W(t)\mu_W(t + \tau) = \alpha \min(t, t + \tau), \\ \text{Cov}[W(2), W(5)] &= 0.03 \min(2, 5) = 0.06. \end{aligned}$$

The samples are not independent: covariance is nonzero. Only the increments over nonoverlapping intervals are independent. Also, $\text{Var}[W(t)] = C_W(t, 0) = \alpha t$.

Question 5: Seasonal temperatures: means, covariance and correlation.

(a) Write $b_n = 16[1 - \cos(2\pi n/365)]$, so $C_n = b_n + 4X_n$, with $E[X_n] = 0$ and $\text{Var}[X_n] = 1$.

$$\begin{aligned}\mu_C[n] &= b_n + 4E[X_n] = b_n, & \text{Var}[C_n] &= 4^2 \text{Var}[X_n] = 16. \\ E[C_n^2] &= \text{Var}[C_n] + \mu_C[n]^2 = 16 + b_n^2 \\ &= 16 + 256 \left[1 - \cos\left(\frac{2\pi n}{365}\right) \right]^2.\end{aligned}$$

(b) Use the lag convention, with $1 \leq m, m+k \leq 365$. Deterministic means drop out of covariance. At $k = 0$, the indices refer to the *same* random variable, so

$$\text{Cov}[X_m, X_{m+0}] = \text{Cov}[X_m, X_m] = \text{Var}[X_m] = 1.$$

At $k \neq 0$, the two variables are independent, so their covariance is zero. Including the scale factor $4^2 = 16$ gives

$$\begin{aligned}C_C[m, k] &= \text{Cov}[C_m, C_{m+k}] = 16 \text{Cov}[X_m, X_{m+k}] \\ &= \begin{cases} 16, & k = 0, \\ 0, & k \neq 0, \end{cases} \\ R_C[m, k] &= C_C[m, k] + \mu_C[m]\mu_C[m+k] = \begin{cases} 16 + b_m^2, & k = 0, \\ b_m b_{m+k}, & k \neq 0. \end{cases}\end{aligned}$$

At nonzero lag the random fluctuations are independent and have zero covariance. Autocorrelation is the *uncentered* product expectation, so the nonzero deterministic means still contribute $b_m b_{m+k}$.