

Tutorial Test 9 - Tutoriaal Toets 9 (20 minutes)

Name and surname / Naam en van	Memo	Mark / Punt [12]
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Mini formula sheet / Mini-formuleblad

$\phi_X(s) = E[e^{sX}] = \sum_{x \in S_X} e^{sx} P_X(x)$	$E[X^k] = \phi_X^{(k)}(0)$ $\text{Var}[X] = E[X^2] - (E[X])^2$	$M_n = \frac{1}{n} \sum_{i=1}^n X_i, \quad X_i \text{ iid/oiv}$ $\frac{M_n - \mu_X}{\sigma_X / \sqrt{n}} \sim \text{Gaussian}(\mu_X = 0, \sigma_X = 1)$ $\text{large } n / \text{groot } n, \quad 0 < \sigma_X^2 < \infty$
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The error X , in millivolts, in one reading of a digital sensor has support $S_X = \{-1, 0, 2\}$ and PMF

x	-1	0	2
$P_X(x)$	1/4	1/2	1/4

To reduce random fluctuations, the sensor averages 64 readings. Assume their errors $X_1, \dots, X_{64}$ are independent and identically distributed copies of X . The average error is

$$M_{64} = \frac{1}{64} \sum_{i=1}^{64} X_i.$$

You may use $\Phi(1.84) = 0.9671$, where Φ is the standard Gaussian CDF.

- a) Determine the moment-generating function $\phi_X(s)$. [3]
- b) Use derivatives of $\phi_X(s)$ to calculate the numerical values of $E[X]$ and $\text{Var}[X]$. [4]
- c) Give the Gaussian approximation to the distribution of M_{64} , specifying the mean and standard deviation as the two parameters in $\text{Gaussian}(\mu, \sigma)$. [2]
- d) Use this Gaussian approximation directly to estimate $P(0 \leq M_{64} \leq 0.5)$. [3]

Die fout X , in millivolt, in een lesing van 'n digitale sensor het steun $S_X = \{-1, 0, 2\}$ en WMF

x	-1	0	2
$P_X(x)$	1/4	1/2	1/4

Om ewekansige skommelings te verminder, neem die sensor die gemiddelde van 64 lesings. Aanvaar dat hul foute $X_1, \dots, X_{64}$ onafhanklike en identies verdeelde kopieë van X is. Die gemiddelde fout is

$$M_{64} = \frac{1}{64} \sum_{i=1}^{64} X_i.$$

Jy mag $\Phi(1.84) = 0.9671$ gebruik, waar Φ die standaard-Gaussiese KVF is.

- a) Bepaal die momentvoortbringende funksie $\phi_X(s)$. [3]
- b) Gebruik afgeleides van $\phi_X(s)$ om die numeriese waardes van $E[X]$ en $\text{Var}[X]$ te bereken. [4]
- c) Gee die Gaussiese benadering tot die verdeling van M_{64} , deur die gemiddelde en standaardafwyking as die twee parameters in $\text{Gaussian}(\mu, \sigma)$ te spesifiseer. [2]
- d) Gebruik hierdie Gaussiese benadering direk om $P(0 \leq M_{64} \leq 0.5)$ te beraam. [3]

a)
$$\begin{aligned} \phi_X(s) &= \sum_{x \in S_X} e^{sx} P_X(x) \\ &= e^{-s} \left(\frac{1}{4}\right) + e^0 \left(\frac{1}{2}\right) + e^{2s} \left(\frac{1}{4}\right) \\ &= \frac{e^{-s}}{4} + \frac{1}{2} + \frac{e^{2s}}{4} \end{aligned}$$

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✓
✓

$$b) E[X] = \phi'_x(0) = -\frac{e^{-s}}{4} + 0 + \frac{2e^{2s}}{4} \bigg|_{s=0} \quad \checkmark \text{ derivative + } s=0 \text{ sub.}$$

$$= -\frac{1}{4} + \frac{2}{4} = \frac{1}{4} \quad \checkmark$$

$$E[X^2] = \phi''_x(0) = \frac{e^{-s}}{4} + \frac{4e^{2s}}{4} \bigg|_{s=0}$$

$$= \frac{1}{4} + 1 = \frac{5}{4} \quad \checkmark$$

$$\text{Var}[X] = E[X^2] - E[X]^2$$

$$= \frac{5}{4} - \left(\frac{1}{4}\right)^2 = \frac{19}{16} \quad \checkmark$$

$$c) \mu = E[M_{64}] = E[X] = \frac{1}{4} \quad \checkmark$$

$$\sigma = \text{Var}[M_{64}] = \frac{\sigma_x}{\sqrt{n}} = \frac{\sqrt{\frac{19}{16}}}{\sqrt{64}} = \frac{\sqrt{19}}{32} = 0,1362... \quad \checkmark$$

$$d) P(0 \leq M_{64} \leq 0,5) = \Phi\left(\frac{0,5 - \frac{1}{4}}{\frac{\sqrt{19}}{32}}\right) - \Phi\left(\frac{0 - \frac{1}{4}}{\frac{\sqrt{19}}{32}}\right) \quad \checkmark$$

$$= \Phi(\underbrace{1,84}_{\text{rounded}}) - \Phi(\underbrace{-1,84}) = 1 - \Phi(1,84)$$

$$= 2\Phi(1,84) - 1$$

$$= 2(0,9671) - 1$$

$$= 0,9342 \quad \checkmark$$

(true answer around 0,9335 if not rounded)

↓ Process $\checkmark$