

Tutorial Test 8 - Tutoriaal Toets 8 ¹⁸ (~~10~~ minutes)

Name and surname / Naam en van	Memo	Mark / Punt [11]
Student number / Studentenommer		11

Mini formula sheet / Mini-formuleblad		
$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ $\boldsymbol{\mu}_Y = \mathbf{A}\boldsymbol{\mu}_X + \mathbf{b}$	$\mathbf{C}_Y = \mathbf{A}\mathbf{C}_X\mathbf{A}^T$	$z = \frac{x - \mu}{\sigma}$ $\Phi(z) = P(Z \leq z)$ $\mu_Z = 0, \quad \sigma_Z = 1$

Standard Gaussian distribution / Standaard Gaussiese verdeling

z	0.25	0.50	0.60	0.75	1.00	1.25	1.30	1.50	1.75	2.00	2.25	2.33	2.50	2.75	3.00
$\Phi(z)$	0.599	0.691	0.726	0.773	0.841	0.894	0.903	0.933	0.960	0.977	0.988	0.990	0.994	0.997	0.999

Two sensor errors, measured in the same units, form a jointly Gaussian random vector $\mathbf{X} = [X_1 \ X_2]^T$ with the following mean vector and covariance matrix:

$$\boldsymbol{\mu}_X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{C}_X = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

A processor forms a sum channel and a difference channel, with fixed offsets:

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \mathbf{A}\mathbf{X} + \mathbf{b},$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

- Calculate the mean vector $\boldsymbol{\mu}_Y$. [2]
- Calculate the covariance matrix $\mathbf{C}_Y$. Show your working. [3]
- What is the distribution family of the new random vector $\mathbf{Y}$? Based on this and your result in (b), discuss whether Y_1 and Y_2 are independent. [2]
- State the distribution family of Y_2 and its parameters (no PDF required). Then calculate $P(Y_2 > 6)$ using the Z table above. [4]

Twee sensorfoute, gemeet in dieselfde eenhede, vorm 'n gesamentlik Gaussiese toevalsvektor $\mathbf{X} = [X_1 \ X_2]^T$ met die volgende gemiddeldevektor en kovariansiematriks:

$$\boldsymbol{\mu}_X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{C}_X = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

'n Verwerker vorm 'n somkanaal en 'n verskilkanaal, met vaste verskuiwings:

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \mathbf{A}\mathbf{X} + \mathbf{b},$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

- Bereken die gemiddelde vektor $\boldsymbol{\mu}_Y$. [2] 2
- Bereken die kovariansiematriks $\mathbf{C}_Y$. Toon jou berekening. [3] 3
- Wat is die verdelingsfamilie van die nuwe toevalsvektor $\mathbf{Y}$? Bespreek, op grond hiervan en jou antwoord in (b), of Y_1 en Y_2 onafhanklik is. [2] 2
- Gee die verdelingsfamilie van Y_2 en die parameters daarvan (geen PDF word vereis nie). Bereken dan $P(Y_2 > 6)$ met die Z-tabel hierbo. [4] 4

$$\begin{aligned}
 a) \quad \bar{\boldsymbol{\mu}}_Y &= \mathbf{A} \bar{\boldsymbol{\mu}}_X + \bar{\mathbf{b}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \checkmark \text{ Process.} \\
 &= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \checkmark \checkmark
 \end{aligned}$$

$$b) C_Y = A C_X A^T$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 0 \\ 0 & 4 \end{bmatrix}$$

✓ Process.

c) $\bar{Y}$ is Bivariate Gaussian ✓ because it is an affine transform of Gaussian random vector $\bar{X}$.

Because $C_Y(1,2) = \text{Cov}[Y_1, Y_2] = 0$ Y_1 and Y_2 are independent. ✓ with correct reason.

d) Y_2 is Gaussian ($\mu_{Y_2} = 4$, $\sigma_{Y_2}^2 = 4$ or $\sigma_{Y_2} = 2$) ✓

$$\begin{aligned} P(Y_2 > 6) &= 1 - P(Y_2 \leq 6) \\ &= 1 - \Phi\left(\frac{6 - \mu_{Y_2}}{\sigma_{Y_2}}\right) \\ &= 1 - \Phi\left(\frac{6 - 4}{2}\right) \checkmark \\ &= 1 - \Phi\left(\frac{2}{2}\right) \\ &= 1 - \Phi(1) \\ &= 1 - 0,841 \\ &= 0,159 \checkmark \end{aligned}$$