Systems and Signals 344

2025

Lecture 33

Power Density Spectrum

Not in textbook - Supplementary on book website

Instantaneous Power Dissipation

If $X(t)$ is a wide sense stationary process then the **average power** is

$$R_X(0) = \mathrm{E}[X^2(t)] \text{ or } R_X[0] = E[X_n]$$

- The definition of the **instantaneous power dissipation**
$$P(t) = V(t)I(t) = \frac{V^2(t)}{R} = I^2(t)R.$$
- If we make the resistance $R = 1\Omega$, and the variable being measured (either voltage or current) $x(t)$, then the instantaneous power is $P(t) = x^2(t)$.
- By extension, we refer to the random variable $X^2(t)$ as the instantaneous power of the process $X(t)$.
- The **Energy** consumed up to time t_1 is:

$$E(t_1) = \int_{-\infty}^{t_1} P(t)dt = \int_{-\infty}^{t_1} x^2(t)dt$$

Parseval's Theorems

Parseval's Theorem relates *total energy* in time and frequency domains.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2\,d\omega=\int_{-\infty}^{\infty}|X(2\pi f)|^2\,df=\int_{-\infty}^{\infty}|x(t)|^2\,dt$$

Where $X(\omega)$ represents the **continuous Fourier transform**.

Plancherel's Theorem is a more general formulation of this, using both $X(\omega)$ and $Y(\omega)$ which relates the total (cross) energy of a signal in time and frequency domains.

Power Density Spectrum

- The autocorrelation function of stochastic process ($R_X(\tau)$) conveys information about the *time structure* of the process.
  - If $X(t)$ is *stationary* and $R_X(\tau)$ decreases rapidly with increasing τ, it is likely that a sample function of $X(t)$ will change abruptly in a short time.
  - Conversely, if the decline in $R_X(\tau)$ with increasing τ is gradual, it is likely that a sample function of $X(t)$ will be slowly time-varying.
- **Fourier transforms** offer another view of the variability of functions of time.
  - A rapidly varying function of time has a Fourier transform with high magnitudes at high frequencies, and a slowly varying function has a Fourier transform with low magnitudes at high frequencies.

Power Density Spectrum

- In the study of stochastic processes, the **power spectral density function**, $S_X(f)$, provides a *frequency-domain representation* of the *time structure* of stochastic process $X(t)$.
- To present the definition of $S_X(f)$ formally, we first establish our notation for the Fourier transform as a function of the frequency variable f Hz.

Fourier Transform Functions $g(t)$ and $G(f)$ are a Fourier transform pair if

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt \quad \text{(FT)} \qquad\qquad g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df \quad \text{(IFT)}$$

Time function	Fourier Transform
$\delta(\tau)$	1
1	$\delta(f)$
$\delta(\tau - \tau_0)$	$e^{-j2\pi f\tau_0}$
$u(\tau)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$e^{j2\pi f_0\tau}$	$\delta(f - f_0)$
$\cos 2\pi f_0\tau$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin 2\pi f_0\tau$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$
$ae^{-a\tau}u(\tau)$	$\frac{a}{a + j2\pi f}$
$ae^{-a\|\tau\|}$	$\frac{2a^2}{a^2 + (2\pi f)^2}$
$ae^{-\pi a^2\tau^2}$	$e^{-\pi f^2/a^2}$
$\mathrm{rect}(\tau/T)$	$T\,\mathrm{sinc}(fT)$
$\mathrm{sinc}(2W\tau)$	$\frac{1}{2W}\,\mathrm{rect}(\frac{f}{2W})$
$g(\tau - \tau_0)$	$G(f)e^{-j2\pi f\tau_0}$
$g(\tau)e^{j2\pi f_0\tau}$	$G(f - f_0)$
$g(-\tau)$	$G^*(f)$
$\frac{dg(\tau)}{d\tau}$	$j2\pi f G(f)$
$\int_{-\infty}^{\tau} g(v)\,dv$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$
$\int_{-\infty}^{\infty} h(v)g(\tau - v)\,dv$	$G(f)H(f)$
$g(t)h(t)$	$\int_{-\infty}^{\infty} H(f')G(f - f')\,df'$

Note that a is a positive constant and that the rectangle and sinc functions are defined as

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| < 1/2, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

Power Density Spectrum

- Recall that not all functions of time have Fourier transforms.
- For many functions that extend over *infinite time*, the time integral does not exist.
- Sample functions $x(t)$ of a *stationary stochastic process* $X(t)$ are usually of this nature.
- To work with these functions in the frequency domain, we begin with $x_T(t)$, a *truncated* version of $x(t)$.
  - It is identical to $x(t)$ for $-T \leq t \leq T$ and 0 elsewhere.
  - We use the notation $X_T(f)$ for the Fourier transform of this function.

$$X_T(f) = \int_{-T}^{T} x(t) e^{-j2\pi ft} \, dt$$

- $|X_T(f)|^2$, the squared magnitude of $X_T(f)$, appears in the definition of $S_X(f)$.
- If $x(t)$ is an electrical signal measured in volts or amperes, $|X_T(f)|^2$ has units of energy.
- Its time average, $|X_T(f)|^2/2T$, has units of power.
- $S_X(f)$ is the limit as the time window goes to infinity of the expected value of this function:

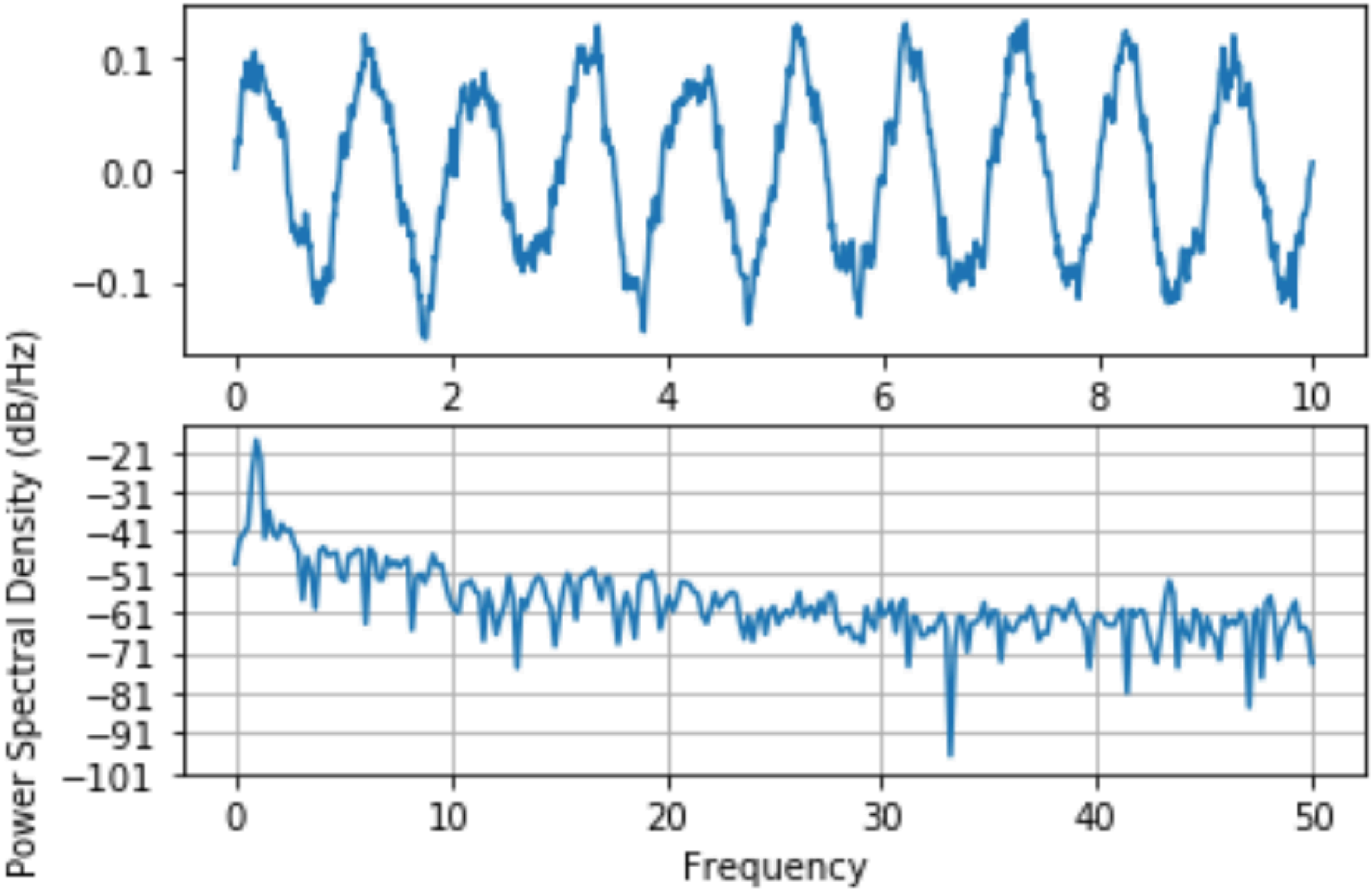
The **power spectral density function** ($S_X(f)$) of the *wide sense stationary stochastic process* $X(t)$ is (using the Fourier Transform)

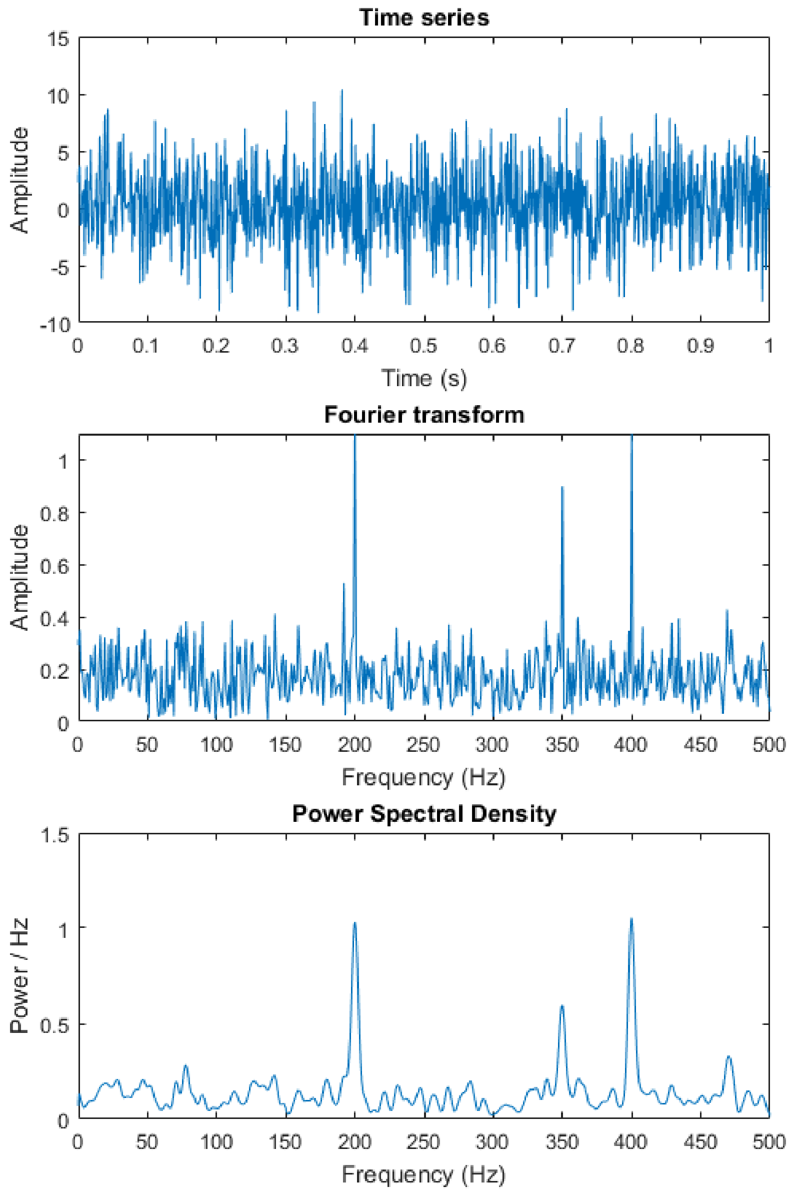
$$S_X(f) = \lim_{T\to\infty} \frac{1}{2T}\mathrm{E}\left[|X_T(f)|^2\right] = \lim_{T\to\infty} \frac{1}{2T}\mathrm{E}\left[\left|\int_{-T}^{T} x(t)e^{-j2\pi ft}, dt\right|^2\right]$$

Power Density Spectrum

$$S_X(f) = \lim_{T\to\infty} \frac{1}{2T} \mathrm{E}\left[|X_T(f)|^2\right] = \lim_{T\to\infty} \frac{1}{2T} \mathrm{E}\left[\left|\int_{-T}^{T} x(t)e^{-j2\pi ft}, dt\right|^2\right]$$

- We refer to $S_X(f)$ as a density function because it can be interpreted as the amount of power in $X(t)$ in the infinitesimal range of frequencies $[f, f + df]$.
  - It therefore states, *where* in frequency most of the power is located.
- Physically, $S_X(f)$ has units of $\mathbf{watts/Hz} = \mathbf{Joules}$.
- That means that the integral of $S_X(f)$ over all frequencies is simply the **average power** of $X(t)$.


0.1
0.0
−0.1
0
2
4
6
8
10
Power Spectral Density (dB/Hz)
−21
−31
−41
−51
−61
−71
−81
−91
−101
0
10
20
30
40
50
Frequency



Time series
Amplitude
15
10
5
0
-5
-10
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time (s)
Fourier transform
Amplitude
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
250
300
350
400
450
500
Frequency (Hz)
Power Spectral Density
Power / Hz
1.5
1
0.5
0
0
50
100
150
200
250
300
350
400
450
500
Frequency (Hz)

Power Density Spectrum

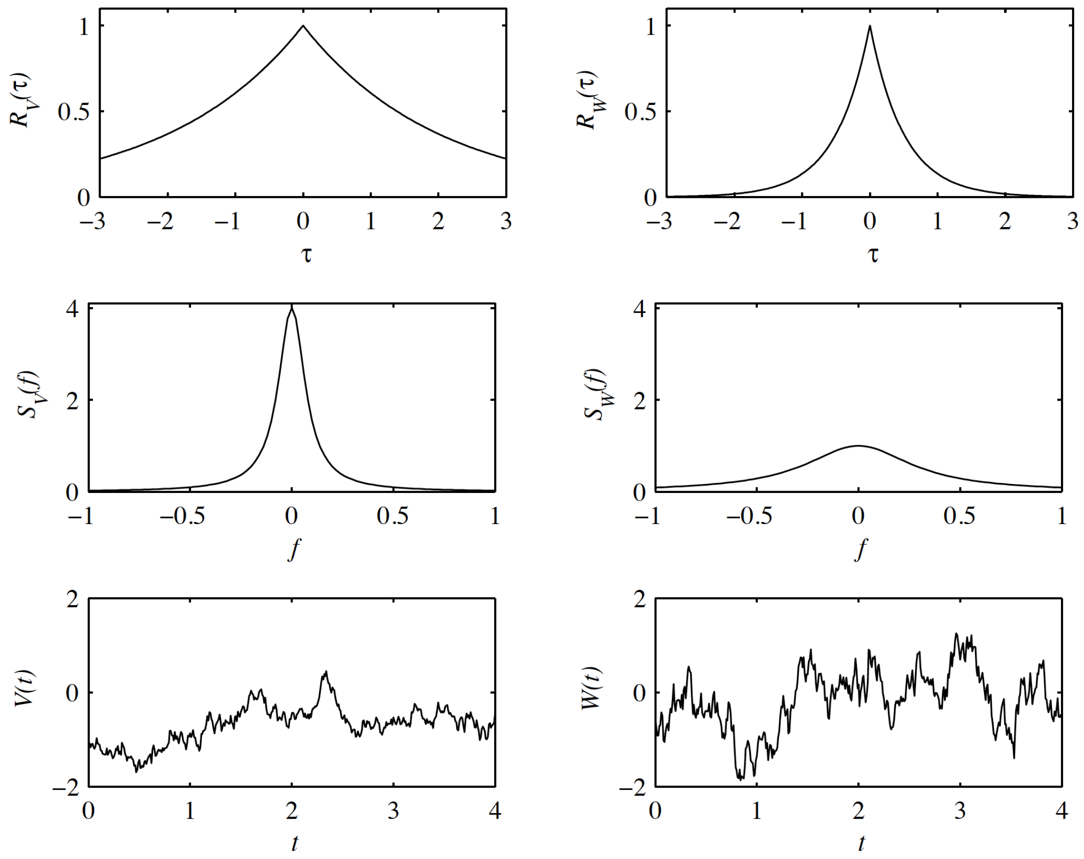
- Both the *autocorrelation function* and the *power spectral density function* convey information about the time structure of $X(t)$. The **Wiener-Khintchine theorem** shows that they convey the same information, which states:

If $X(t)$ is a *wide sense stationary stochastic process*, $R_X(\tau)$ and $S_X(f)$ are the Fourier transform pair:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau, \qquad R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df$$

Power Density Spectrum

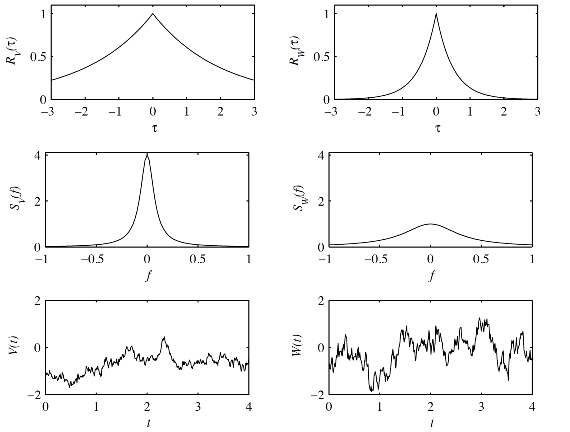
- For a *wide sense stationary random process* $X(t)$, the power spectral density $S_X(f)$ is a real-valued function with the following properties:
  1. $S_X(f) \geq 0 \ \text{ for all } f$
  2. $P_X = \int_{-\infty}^{\infty} S_X(f)\, df = \mathbf{E}[X^2(t)] = R_X(0)$
  3. $S_X(-f) = S_X(f)$ symmetrical



Random processes $V(t)$ and $W(t)$ with autocorrelation functions $R_V(\tau) = e^{-0.5|\tau|}$ and $R_W(\tau) = e^{-2|\tau|}$ are examples of the process $X(t)$ in Example 11.15. These graphs show $R_V(\tau)$ and $R_W(\tau)$, the power spectral density functions $S_V(f)$ and $S_W(f)$, and sample paths of $V(t)$ and $W(t)$.

Slow change - energy in low frequency

Rapid change - energy in high frequency



Random processes $V(t)$ and $W(t)$ with autocorrelation functions $R_V(\tau) = e^{-0.5|\tau|}$ and $R_W(\tau) = e^{-2|\tau|}$ are examples of the process $X(t)$ in Example 11.15. These graphs show $R_V(\tau)$ and $R_W(\tau)$, the power spectral density functions $S_V(f)$ and $S_W(f)$, and sample paths of $V(t)$ and $W(t)$.

- Note, both have average power is $A = R_X(0) = \mathbf{1\ Watt}$
- $W(t)$ has a narrower autocorrelation (less dependence between two values of the process with a given time separation) and a wider power spectral density (more power at higher frequencies) than $V(t)$.
- The sample function $w(t)$ fluctuates more rapidly with time than $v(t)$.

Example 15

A wide sense stationary process $X(t)$ has autocorrelation function $R_X(\tau) = Ae^{-b|\tau|}$ where $b > 0$. Derive the power spectral density function $S_X(f)$ and calculate the average power $\mathrm{E}[X^2(t)]$.

$ae^{-a\tau}u(\tau)$	$\dfrac{a}{a + j2\pi f}$
$ae^{-a\|\tau\|}$	$\dfrac{2a^2}{a^2 + (2\pi f)^2}$
$ae^{-\pi a^2\tau^2}$	$e^{-\pi f^2/a^2}$

Example 2

Consider the following stochastic process $X(t) = \cos(2\pi f_c t + \Theta)$ where Θ is uniformly distributed between 0 and 2π.

Previously we derived $R_X(t,\tau) = \frac{1}{2}\cos(2\pi f_c \tau)$.

Plot $S_X(f)$ and calculate the Average Power.

$e^{j2\pi f_0 \tau}$	$\delta(f - f_0)$
$\cos 2\pi f_0 \tau$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\sin 2\pi f_0 \tau$	$\frac{1}{2j}\delta(f - f_0) - \frac{1}{2j}\delta(f + f_0)$

Cross Spectral Density

When two processes are jointly wide sense stationary, we can study the cross-correlation in the frequency domain.

For *jointly wide sense stationary stochastic processes* $X(t)$ and $Y(t)$, the Fourier transform of the cross-correlation yields the **cross spectral density**

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j2\pi f\tau}\, d\tau$$

We encounter cross-correlations in experiments that involve *noisy* observations of a *wide sense stationary random process* $X(t)$. e.g.

$$Y(t) = X(t) + N(t)$$

Example 20

In Example 13.24, we were interested in $X(t)$ but we could observe only $Y(t) = X(t) + N(t)$ where $N(t)$ is a wide sense stationary noise process with $\mu_N = 0$. In this case, when $X(t)$ and $N(t)$ are jointly wide sense stationary, we found that

$$R_Y(\tau) = R_X(\tau) + R_{XN}(\tau) + R_{NX}(\tau) + R_N(\tau). \tag{108}$$

Find the power spectral density of the output $Y(t)$.

. . . .

By taking a Fourier transform of both sides, we obtain the power spectral density of the observation $Y(t)$.

$$S_Y(f) = S_X(f) + S_{XN}(f) + S_{NX}(f) + S_N(f). \tag{109}$$

Example 21

Continuing Example 20, suppose $N(t)$ and $X(t)$ are independent. Find the autocorrelation and power spectral density of the observation $Y(t)$.

. . . .

In this case,

$$R_{XN}(\tau) = \mathrm{E}[X(t)N(t+\tau)] = \mathrm{E}[X(t)]\,\mathrm{E}[N(t+\tau)] = 0. \tag{110}$$

Similarly, $R_{NX}(\tau) = 0$. This implies

$$R_Y(\tau) = R_X(\tau) + R_N(\tau), \tag{111}$$

and

$$S_Y(f) = S_X(f) + S_N(f). \tag{112}$$