Systems and Signals 344

2025

Lecture 32

Cross-correlation & Gaussian Processes

Sections 13.10-13.11

Recap: Autocovariance and Autocorrelation

The **autocovariance** function of the stochastic process $X(t)$ is

$$C_X(t,\tau) = \mathrm{Cov}[X(t), X(t+\tau)]$$

The **autocorrelation** function of the stochastic process $X(t)$ is

$$R_X(t,\tau) = \mathrm{E}[X(t)X(t+\tau)]$$

The autocorrelation and autocovariance functions of the process $X(t)$ satisfy

$$C_X(t,\tau) = R_X(t,\tau) - \mu_X(t)\mu_X(t+\tau)$$

Example 1

A signal $X(t)$ is noisy periodic signal and is wide sense stationary. Use $R_X(\tau)$ to determine the period of the signal.

Cross-correlation

- In many applications, it is necessary to consider the relationship of *two* stochastic processes $X(t)$ and $Y(t)$.
- For certain experiments, it is appropriate to model $X(t)$ and $Y(t)$ as *independent* processes.
  - In this simple case, any set of random variables $X(t_1), \ldots, X(t_k)$ from the $X(t)$ process is independent of any set of random variables $Y(t'_1), \ldots, Y(t'_j)$ from $Y(t)$.
- In general, however, a complete probability model of two processes consists of a *joint PMF* or a *joint PDF* of all sets of random variables contained in the processes.
- Such a joint probability function completely expresses the relationship of the two processes.
- However, finding and working with such a joint probability function is usually prohibitively difficult.

Cross-correlation

- To obtain useful tools for analyzing a pair of processes, we recall that the **covariance** and the **correlation** of a pair of random variables provide valuable information about the relationship between the random variables.
- To use this information to understand a pair of stochastic processes, we work with the correlation and covariance of the random variables $X(t)$ and $Y(t+\tau)$.

The **cross-correlation** of *random processes* $X(t)$ and $Y(t)$ is

$$R_{XY}(t,\tau) = \mathrm{E}[X(t)Y(t+\tau)]$$

and of *random sequences* X_n and Y_n is

$$R_{XY}[m,k] = \mathrm{E}[X_m Y_{m+k}]$$

Just as for the autocorrelation, there are many interesting practical applications in which the cross-correlation depends only on one time variable, namely the *time difference* τ or the *index difference* k. Then:

For *random processes* $X(t)$ and $Y(t)$ are **jointly wide sense stationary** if $X(t)$ and $Y(t)$ are both wide sense stationary, *and* the cross-correlation depends only on the time difference between the two random variables:

$$R_{XY}(t,\tau) = R_{XY}(\tau)$$

similar for *random sequences* X_n and Y_n where the cross-correlation depends only on the index difference:

$$R_{XY}[m,k] = R_{XY}[k]$$

Recap: Wide Sense Stationary Stochastic Processes

For a wide sense stationary process $X(t)$, the autocorrelation function $R_X(\tau)$ has the following properties:

$$R_X(0) \geq 0, \quad R_X(\tau) = R_X(-\tau), \quad R_X(0) \geq |R_X(\tau)|$$

If X_n is a wide sense stationary random sequence:

$$R_X[0] \geq 0, \quad R_X[k] = R_X[-k], \quad R_X[0] \geq |R_X[k]|$$

This indicates that the autocorrelation of a wide sense stationary process $X(t)$ is symmetric about $\boldsymbol{\tau = 0}$ (continuous-time) or $\boldsymbol{k = 0}$ (random sequence).

Cross-correlation

The cross-correlation of jointly wide sense stationary processes has a corresponding symmetry.

If $X(t)$ and $Y(t)$ are **jointly wide sense stationary** continuous-time processes, then

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

or if X_n and Y_n are **jointly wide sense stationary** random sequences, then

$$R_{XY}[k] = R_{YX}[-k]$$

Example 2

Radar transmitted signals is modeled with $X(t)$ and received signal with $Y(t)$. They are jointly wide sense stationary. Use $R_{XY}(\tau)$ to determine the distance an object is away.

Cross-correlation

Example 13.18

Suppose we are interested in $X(t)$ but we can observe only

$$Y(t) = X(t) + N(t), \tag{13.56}$$

where $N(t)$ is a noise process that interferes with our observation of $X(t)$. Assume $X(t)$ and $N(t)$ are independent wide sense stationary processes with $\mathrm{E}[X(t)] = \mu_X$ and $\mathrm{E}[N(t)] = \mu_N = 0$. Is $Y(t)$ wide sense stationary? Are $X(t)$ and $Y(t)$ jointly wide sense stationary? Are $Y(t)$ and $N(t)$ jointly wide sense stationary?

Gaussian Processes

- The *central limit theorem* helps explain the proliferation of Gaussian random variables in nature. The same insight extends to Gaussian stochastic processes.
- For electrical and computer engineers, the noise voltage in a resistor is a pervasive example of a phenomenon that is accurately modeled as a **Gaussian stochastic process**.
- In a Gaussian process, every collection of sample values is a *Gaussian random vector*.

Gaussian Processes

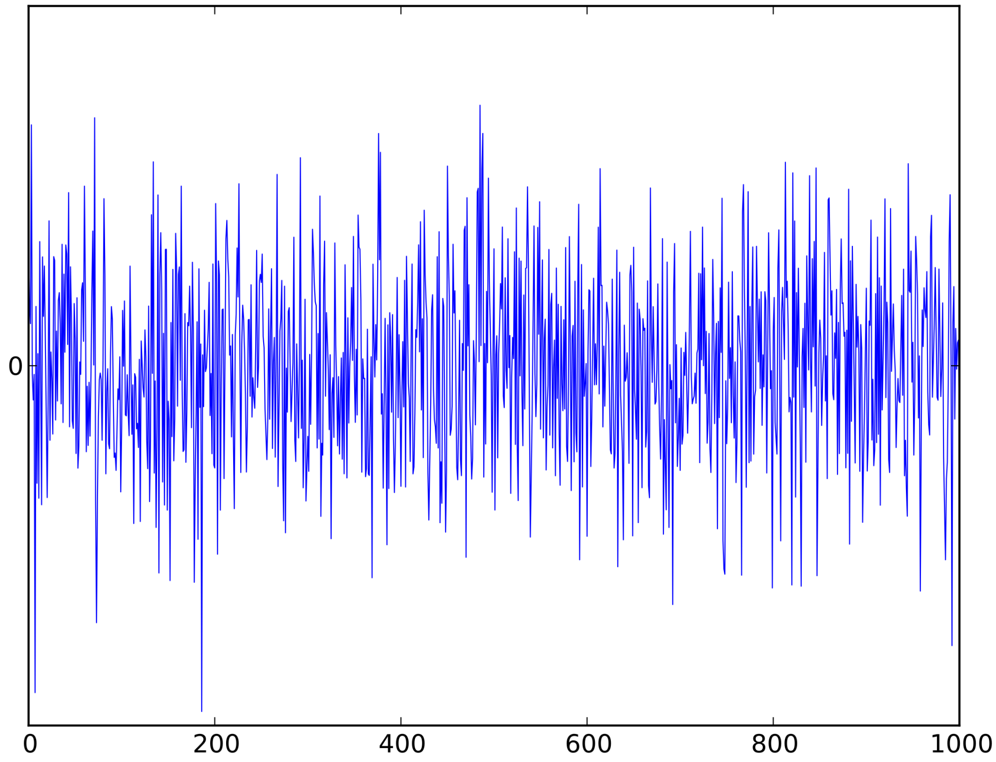
$X(t)$ is a **Gaussian stochastic process** if and only if $\mathbf{X} = [X(t_1) \quad \cdots \quad X(t_k)]'$ is a *Gaussian random vector* for any integer $k > 0$ and any set of time instants $t_1, t_2, \ldots, t_k$

Similar for X_n is a **Gaussian random sequence** if and only $\mathbf{X} = [X_{n_1} \quad \cdots \quad X_{n_k}]'$ is a *Gaussian random vector* for any integer $k > 0$ and any set of time instants $n_1, n_2, \ldots, n_k$

Gaussian Processes

- The *Brownian motion process* in is a special case of a Gaussian process.
- Although the Brownian motion process is *not* stationary, our primary interest will be in *wide sense stationary Gaussian processes*.
- In this case, the probability model for the process is completely specified by the expected value μ_X and the autocorrelation function $R_X(\tau)$ or $R_X[k]$.
- As a consequence, if $X(t)$ (or X_n) is a wide sense stationary Gaussian process (or sequence), then $X(t)$ is a stationary Gaussian process (or sequence), since the Gaussian PDF only relies on Covariance and the Expected value, which are time invariant.

White Gaussian Noise Process



White Gaussian Noise Process

$W(t)$ is a **white Gaussian noise process** if and only if $W(t)$ is a *stationary* Gaussian stochastic process with the properties $\mu_W = 0$ and $R_W(\tau) = \eta_0 \delta(\tau)$, with η_0 indicating the *power level*.

- Where $\delta(\tau)$ is an impulse function: $\delta(\tau) = \begin{cases} \infty & \text{when } \tau = 0 \\ 0 & \text{otherwise} \end{cases}$

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- A consequence of the definition is that for any collection of distinct time instants $t_1, \ldots, t_k$, then $W(t_1), \ldots, W(t_k)$ is a set of *independent* Gaussian random variables.
  - In this case, the value of the noise at time t_i tells nothing about the value of the noise at time t_j.

White Gaussian Noise Process

- While the white Gaussian noise process is a useful mathematical model, it does not conform to any signal that can be observed physically.
- Note that the average noise power is

$$\mathbf{E}[W^2(t)] = R_W(0) = \infty$$

- That is, white noise has infinite average power, which is physically impossible.
- The model is useful, however, because any Gaussian noise signal observed in practice can be interpreted as a filtered white Gaussian noise signal with finite power.

White Gaussian Noise Process

X_n is a *discrete-time* **white Gaussian noise process** with mean μ_X and variance σ_X^2 if and only if X_n is *stationary* random sequence and has

$$\mathrm{E}[X_n] = \mu_X \text{ and } R_X[k] = \sigma_X^2 \delta[k]$$

- Where $\delta[k]$ is the discrete impulse function (Kronecker delta):

$$\delta[k] = \begin{cases} 1 & \text{when } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

The amplitude is not infinite because the discrete-time noise only exists at specific time steps, so we no longer need to worry about infinite power.

Note that the characteristic of *independence* of samples is only a property of White Gaussian Noise, **not** of White Noise in general

Example 3

The moving average of a white Gaussian noise process is

$$Y_n = X_n + 0.5X_{n-1}$$

Where X_n is a white Gaussians noise process with $\sigma_X^2 = 1$.

Determine $R_Y[k]$