Systems and Signals 344

2025

Lecture 31

Stationary Processes & Wide Sense Stationary Stochastic Processes

Sections 13.8-13.9

Recap: Autocovariance and Autocorrelation

The **autocovariance** function of the stochastic process $X(t)$ is

$$C_X(t, \tau) = \mathrm{Cov}[X(t), X(t+\tau)]$$

The **autocorrelation** function of the stochastic process $X(t)$ is

$$R_X(t, \tau) = \mathrm{E}[X(t)X(t+\tau)]$$

The autocorrelation and autocovariance functions of the process $X(t)$ satisfy

$$C_X(t, \tau) = R_X(t, \tau) - \mu_X(t)\mu_X(t+\tau)$$

Stationary Processes

> *A stochastic process is stationary if the probability model does not vary with time.*

A stochastic process is **stationary** if *all* its statistical properties are *time invariant* (do not vary with time), i.e. they are not a function of time

Stationary Processes

- Recall that in a stochastic process, $X(t)$, there is a random variable $X(t_1)$ at every time instant t_1 with PDF $f_{X(t_1)}(x)$.
- For most random processes, the PDF $f_{X(t_1)}(x)$ depends on t_1.
    - For example, when we make daily temperature readings, we expect that readings taken in the winter will be lower than temperatures recorded in the summer.
- However, for a special class of random processes known as stationary processes, $f_{X(t_1)}(x)$ does *not* depend on t_1. That is, for any two time instants t_1 and $t_1 + \tau$,

$$f_{X(t_1)}(x) = f_{X(t_1+\tau)}(x) = f_X(x)$$

- Therefore, in a stationary process, we observe the same random variable at all time instants.

A stochastic process $X(t)$ is **stationary** if and only if for all sets of time instants $t_1, \ldots, t_m$, and any time difference τ,

$$f_{X(t_1),\ldots,X(t_m)}(x_1, \ldots, x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1, \ldots, x_m)$$

A random sequence X_n is **stationary** if and only if for any set of integer time instants $n_1, \ldots, n_m$, and integer time difference k,

$$f_{X_{n_1},\ldots,X_{n_m}}(x_1, \ldots, x_m) = f_{X_{n_{1+k}},\ldots,X_{n_{m+k}}}(x_1, \ldots, x_m)$$

Sometimes, this is called **strict sense stationary**.

Generally it is not obvious whether a stochastic process is stationary. Usually a stochastic process is not stationary. However, proving or disproving stationarity can be tricky (since this must be done for all possible values of τ or k).

Example: The Brownian Motion Process

A **Brownian motion process** $W(t)$ has the property that $W(0) = 0$, and for $\tau > 0$, $W(t+\tau) - W(t)$ is a **Gaussian** $(0, \sqrt{\alpha\tau})$ random variable that is *independent* of $W(t')$ for all $t' \leq t$.

Note that τ is any time period, and α is the **Brownian Motion Parameter**, which indicates how much of a jump from the previous step can be expected.

Is the Brownian Motion Process stationary?

Stationary Processes

If we modify one *stationary* stochastic process to produce a new process. Then:

Let $X(t)$ be a stationary stochastic process. For constants $a > 0$ and b, $Y(t) = aX(t) + b$ is also a stationary stochastic process.

A linear transformation of a stationary process produces another stationary process.

Stationary Processes

- There are many consequences of the time-invariant nature of a stationary random process.
- From the definition of a stationary process $f_{X(t_1)}(x) = f_{X(t_1+\tau)}(x) = f_X(x)$. This implies that:
  - $\mathrm{E}[X(t)] = \mathbf{Constant}$
  - $C_X(t, \tau)$ (autocovariance) and $R_X(t, \tau)$ (autocorrelation) are independent of t and *depend only* on the time difference τ, hence we can write $C_X(\tau)$ and $R_X(\tau)$

Stationary Processes

For a **stationary process** $X(t)$, the expected value, the autocorrelation, and the autocovariance have the following properties for all t:

1. $\mu_X(t) = \mu_X$
2. $R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$
3. $C_X(t,\tau) = R_X(\tau) - {\mu^2}_X = C_X(\tau)$

For a **stationary random sequence** X_n the expected value, the autocorrelation, and the autocovariance satisfy for all n

1. $\mathrm{E}[X_n] = \mu_X$
2. $R_X[n,k] = R_X[0,k] = R_X[k]$
3. $C_X[n,k] = R_X[k] - {\mu^2}_X = C_X[k]$

Stationary Processes

Example 13.16

At the receiver of an AM radio, the received signal contains a cosine carrier signal at the carrier frequency f_c with a random phase Θ that is a sample value of the uniform $(0, 2\pi)$ random variable. The received carrier signal is

$$X(t) = A\cos(2\pi f_c t + \Theta). \tag{13.39}$$

What are the expected value and autocorrelation of the process $X(t)$?

Wide Sense Stationary Stochastic Processes

*A stochastic process is **wide sense stationary** if the expected value is constant with time and the autocorrelation depends only on the time difference between two random variables.*

*A wide sense stationary process is **ergodic** if expected values such as $\mathrm{E}[X(t)]$ and $\mathrm{E}[X^2(t)]$ are equal to corresponding time averages.*

Wide Sense Stationary Stochastic Processes

- There are many applications of probability theory in which investigators do not have a complete probability model of an experiment.
    - Therefore, it is not possible to prove strict sense stationarity.
- Even so, much can be accomplished with partial information about the model.
- Often the partial information takes the form of expected values, variances, correlations, and covariances.
- If the requirements for stationarity hold, based on these parameters alone, without knowing the probability model, we say the process is **wide sense stationary**.
    - This is clearly a less strict requirement.

Wide Sense Stationary Stochastic Processes

$X(t)$ is a **wide sense stationary stochastic process** if and only if $\forall t$,

$$\mathbf{E}[X(t)] = \mu_X, \quad \text{and} \quad R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$

X_n is a **wide sense stationary random sequence** if and only if $\forall n$,

$$\mathbf{E}[X_n] = \mu_X, \quad \text{and} \quad R_X[n,k] = R_X[0,k] = R_X[k]$$

This implies that *every* stationary process or sequence is also wide sense stationary.

Wide Sense Stationary Stochastic Processes

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This implies that *every* stationary process or sequence is also wide sense stationary.

However, if $X(t)$ or X_n is wide sense stationary, it may or may not be stationary. Thus wide sense stationary processes include stationary processes as a subset.

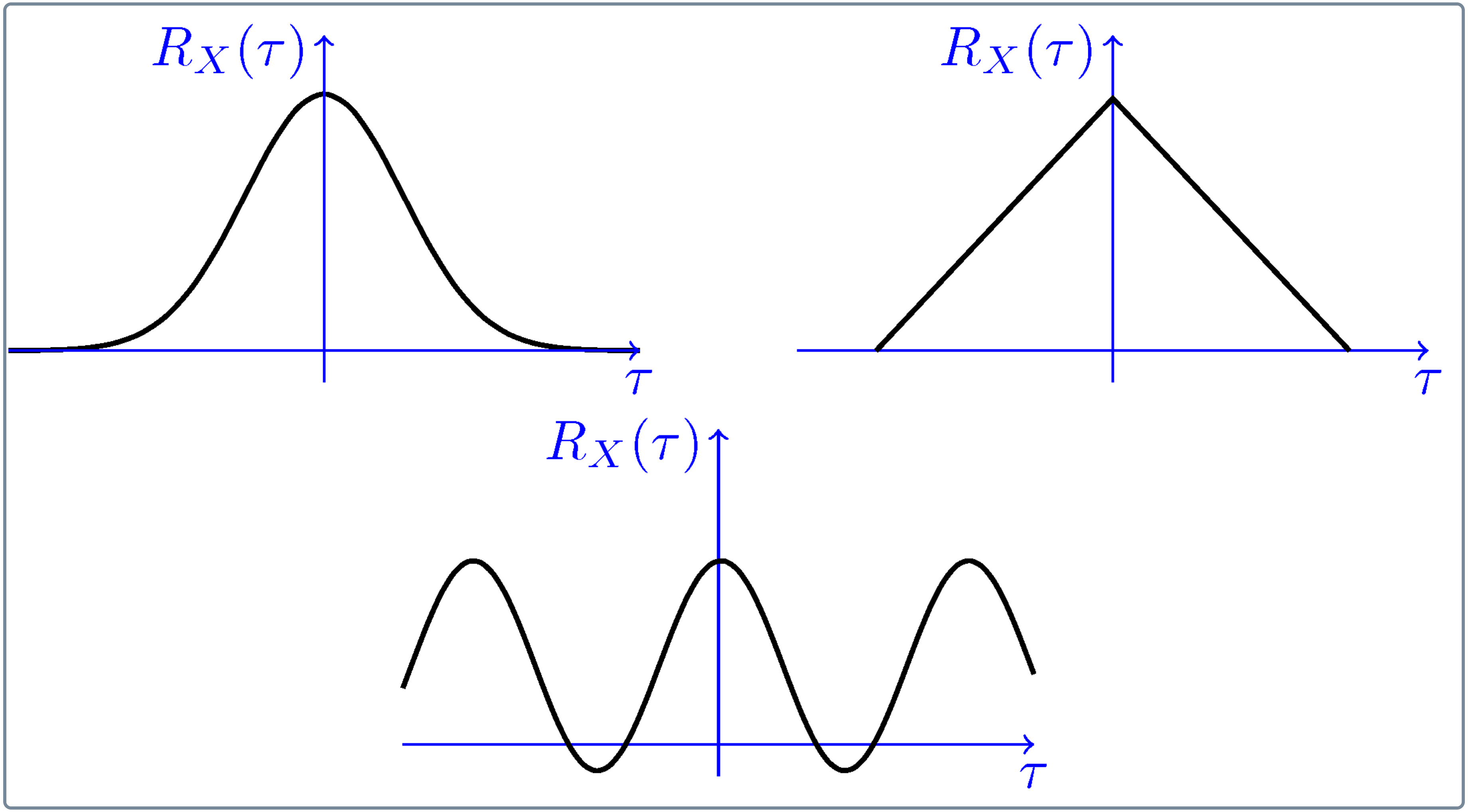
Wide Sense Stationary Stochastic Processes

For a wide sense stationary process $X(t)$, the autocorrelation function $R_X(\tau)$ has the following properties:

$$R_X(0) \geq 0, \quad R_X(\tau) = R_X(-\tau), \quad R_X(0) \geq |R_X(\tau)|$$

If X_n is a wide sense stationary random sequence:

$$R_X[0] \geq 0, \quad R_X[k] = R_X[-k], \quad R_X[0] \geq |R_X[k]|$$


$R_X(\tau)$
τ
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$R_X(0)$ has an important physical interpretation for electrical engineers. It is the **Average Power**.

If $X(t)$ or X_n is a wide sense stationary process then the **average power** (or **expected power**) is $R_X(0) = \mathrm{E}[X^2(t)]$ or $R_X[0] = E[X_n^2]$

If $X(t)$ or X_n is a wide sense stationary process then the **average power** (or **expected power**) is $R_X(0) = \mathrm{E}[X^2(t)]$ or $R_X[0] = E[X_n^2]$

- This comes from the definition of the *instantaneous power dissipation* $P(t) = V(t)I(t) = \dfrac{V^2(t)}{R} = I^2(t)R$.
- If we make the resistance $R = 1\,\Omega$, and the variable being measured (either voltage or current) $x(t)$, then the instantaneous power is $P(t) = x^2(t)$.
- By extension, we refer to the random variable $X^2(t)$ as the instantaneous power of the process $X(t)$.
- Therefore, the above definition uses "average power" for the expected value of the instantaneous power of a process.

Wide Sense Stationary Stochastic Processes

- Recall that we can calculate either *ensemble averages* or *time averages* of stochastic processes.
- In our presentation of stationary processes, we have encountered only ensemble averages including the expected value, the autocorrelation, the autocovariance, and the average power.
- Engineers, however, are accustomed to observing time averages. - For example, if $X(t)$ models a voltage, the time average of sample function $x(t)$ over an interval of duration $2T$ is:

$$\overline{X}(T) = \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

This is the DC voltage of $x(t)$ that is read when you use a multimeter.

Wide Sense Stationary Stochastic Processes

- The relationship of these time averages to the corresponding ensemble averages, μ_X and $E[X^2(t)]$, is a fascinating topic in the study of stochastic processes.
- When $X(t)$ is a *stationary process* such that $\lim_{T\to\infty} \overline{X}(T) = \mu_X$, the process is referred to as **ergodic**.
- In words, for an ergodic process, the *time average* of the sample function of a wide sense stationary stochastic process is equal to the corresponding *ensemble average*.

Wide Sense Stationary Stochastic Processes

- For an electrical signal modeled as a sample function of an *ergodic* process, μ_X and $E[X^2(t)]$ and many other ensemble averages can be observed with familiar measuring equipment.
- For a stationary process $X(t)$, we can view the time average $\overline{X}(T)$ as an *estimate* of the parameter μ_X, analogous to the sample mean $M_n(X)$.
    - The difference, however, is that the sample mean is an average of independent random variables, whereas sample values of the random process $X(t)$ are *correlated*.
    - However, if the autocovariance $C_X(\tau)$ approaches zero quickly, then as T becomes large, most of the sample values have little or no correlation, and we would expect the process $X(t)$ to be ergodic.

Wide Sense Stationary Stochastic Processes

Let $X(t)$ be a stationary random process with expected value μ_X and autocovariance $C_X(\tau)$.

If $\int_{-\infty}^{\infty} |C_X(\tau)| \, d\tau < \infty$, then $\overline{X}(T), \overline{X}(2T), \ldots$ is an *unbiased*, *consistent* sequence of estimates of μ_X.