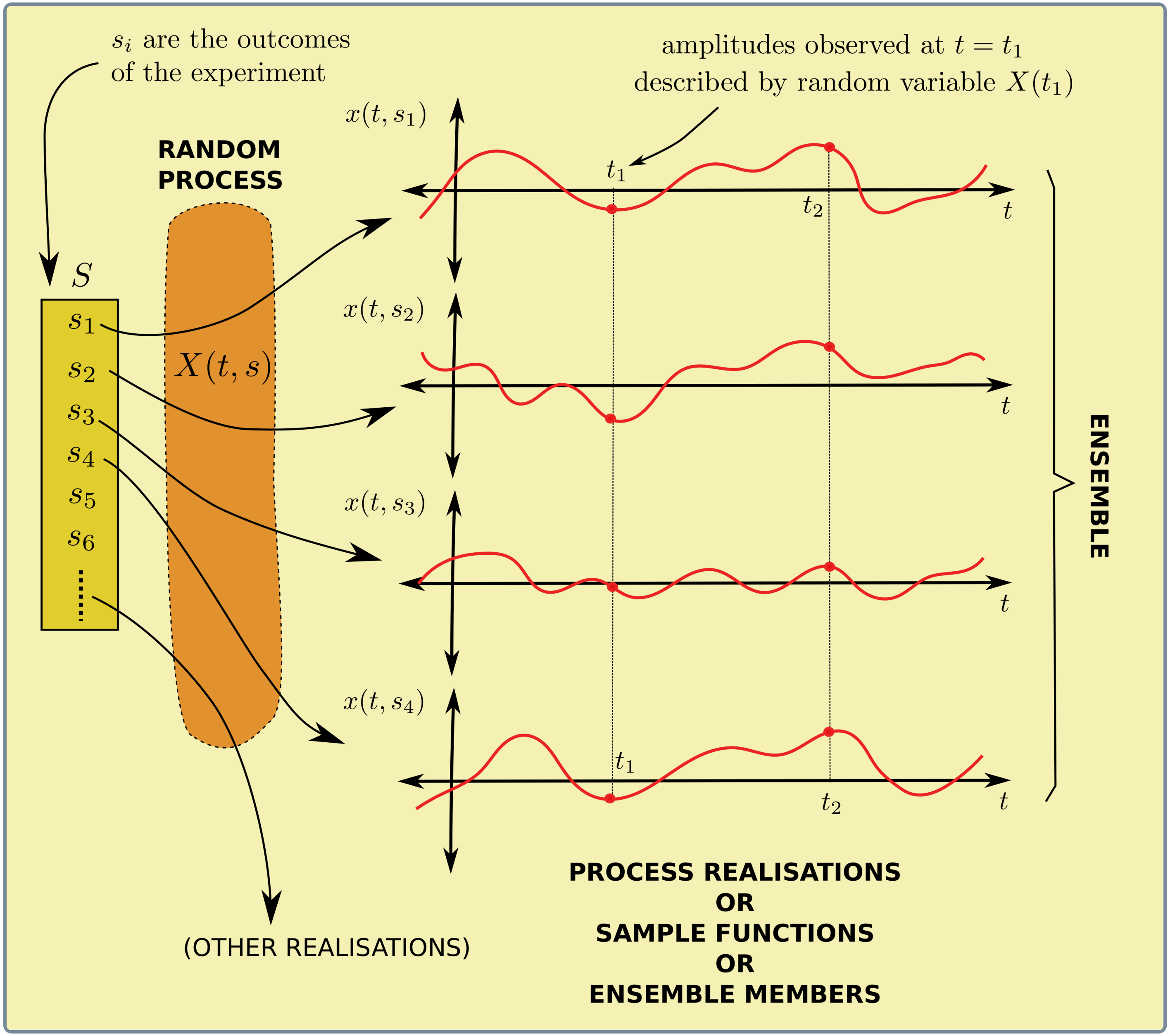
Systems and Signals 344

2025

Lecture 30

The Brownian Motion Process & Expected Value and Correlation & Moments

Sections 13.6-13.7



s_i are the outcomes of the experiment
amplitudes observed at $t = t_1$ described by random variable $X(t_1)$
RANDOM PROCESS
S
s_1
s_2
s_3
s_4
s_5
s_6
$X(t, s)$
$x(t, s_1)$
$x(t, s_2)$
$x(t, s_3)$
$x(t, s_4)$
t_1
t_2
t
ENSEMBLE
(OTHER REALISATIONS)
PROCESS REALISATIONS
OR
SAMPLE FUNCTIONS
OR
ENSEMBLE MEMBERS

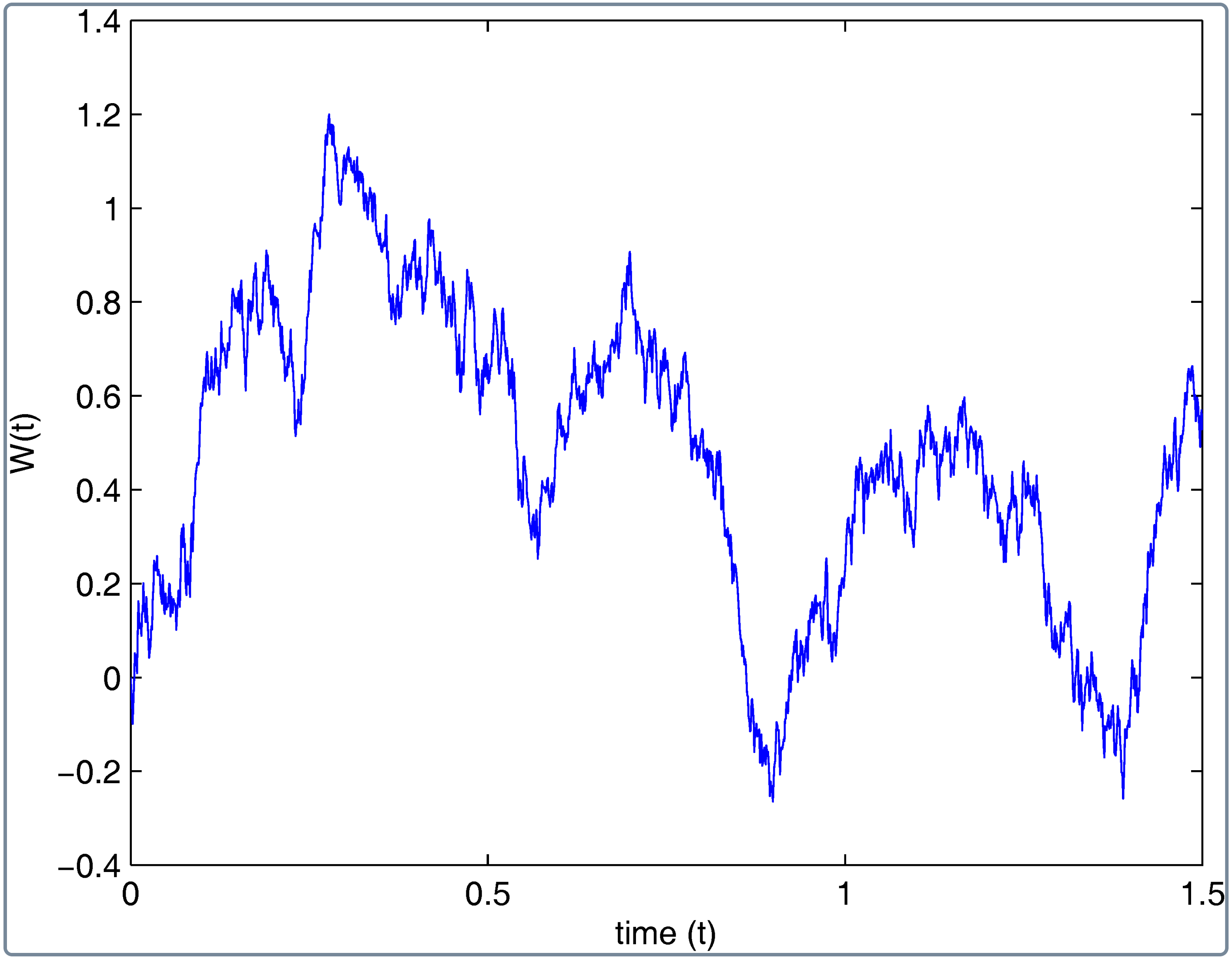
The Brownian Motion Process

*The **Brownian motion process** (or **Wiener Process**) describes a one-dimensional random walk in which at every instant, the position changes by a small increment that is independent of the current position and past history.*

The position change over any time interval is a Gaussian random variable with zero expected value and variance proportional to the time interval.

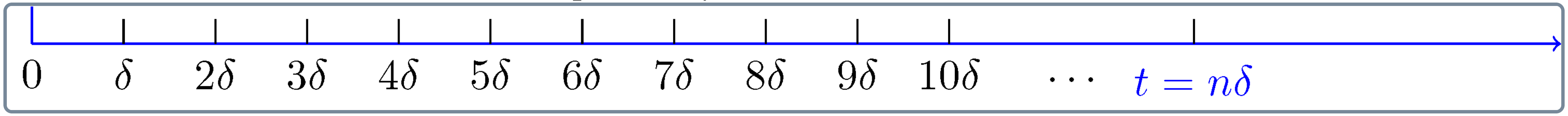
It is a **continuous-time**, **continuous-value** stochastic process.

The Brownian Motion Process



Symmetric Random Walk Process

First we divide a line $[0, \infty)$ into tiny subintervals of length δ.



At each time we flip a fair coin and define the random variable X_i:

$$X_i = \begin{cases} \sqrt{\delta} & p = 0.5, \\ -\sqrt{\delta} & p = 0.5 \end{cases}$$

Thus: $\mathbf{E}[X_i] = 0$ and $\mathbf{Var}[X_i] = \delta$

We we define the stochastic process $W(t) = W(n\delta) = \sum_{i=1} X_i$

Thus: $\mathbf{E}[W(t)] = 0$ and $\mathbf{Var}[W(t)] = t$

The Brownian Motion Process

A **Brownian motion process** $W(t)$ has the property that $W(0) = 0$, and for $\tau > 0$, $W(t + \tau) - W(t)$ is a **Gaussian** $(0, \sqrt{\alpha\tau})$ random variable that is *independent* of $W(t')$ for all $t' \leq t$.

Note that τ is any time period, and α is the **Brownian Motion Parameter**, which indicates how much of a jump from the previous step can be expected.

The Brownian Motion Process

A **Brownian motion process** $W(t)$ has the property that $W(0) = 0$, and for $\tau > 0$, $W(t+\tau) - W(t)$ is a **Gaussian** $(0, \sqrt{\alpha\tau})$ random variable that is *independent* of $W(t')$ for all $t' \leq t$.

Note that: $W(t+\tau) = W(t) + [W(t+\tau) - W(t)]$ where the last term is the "increment" and can be called X and can be *positive or negative*.

From the definition of Brownian Motion the *movement* by an amount X is independent of the *previous position* $W(t)$. This property is called **independent increments**.

Also note that this only explains motion along one axis of motion.

Brownian motion was first described in 1827 by botanist Robert Brown when he was examining the movement of pollen grains in water. It was believed that the movement was the result of the internal processes of the living pollen.

Brown found that the same movement could be observed for any finely ground mineral particles.

In 1905, Albert Einstein identified the source of this movement as random collisions with water molecules in thermal motion.

The theory of Brownian motion was further developed by several distinguished mathematical physicists until Norbert Wiener gave it a rigorous mathematical formulation in his 1918 dissertation and in later papers.

The Brownian Motion Process

- Particle
- Animation

The Brownian Motion Process

For the Brownian motion process $W(t)$, the PDF of $\mathbf{W} = \begin{bmatrix} W(t_1) & \ldots & W(t_k) \end{bmatrix}$ is

$$f_{\mathbf{W}}(\mathbf{w}) = \prod_{n=1} \frac{1}{\sqrt{2\pi\alpha(t_n - t_{n-1})}} e^{-\frac{(w_n - w_{n-1})^2}{2\alpha(t_n - t_{n-1})}}$$

Note that the product is due to the independence of each time interval.

Expected Value and Correlation

*The **expected value** of a stochastic process is a function of time.*

*The **autocovariance** and **autocorrelation** are functions of two time variables. All three functions indicate the rate of change of the sample functions of a stochastic process.*

Expected Value of a Process

- In studying **random variables**, we often refer to *properties* (a single number) of the probability model such as the expected value, the variance, the covariance, and the correlation.
    - These parameters are a few numbers that summarize the complete probability model.

Expected Value of a Process

- In studying **random variables**, we often refer to *properties* (a single number) of the probability model such as the expected value, the variance, the covariance, and the correlation.
  - These parameters are a few numbers that summarize the complete probability model.
- In the case of **stochastic processes**, *deterministic functions of time* provide corresponding summaries of the properties of a complete model.
- Recall that for a stochastic process $X(t)$, $X(t_1)$, the value of the sample functions at time instant t_1, is a random variable.
  - Hence it has a PDF $f_{X(t_1)}(x)$ and expected value $\mathbf{E}[X(t_1)]$.
  - Since $\mathbf{E}[X(t)]$ is simply a number for each value of t, the expected value $\mathbf{E}[X(t)]$ is a *deterministic function of* t.

Expected Value of a Process

The **expected value** of a stochastic process $X(t)$ is the *deterministic function*

$$\mu_X(t) = \mathrm{E}[X(t)]$$

(Note not $\mu_{X(t)}$)

Moments of random processes

Moments about the origin are also just functions of time

of *Stochastic Process* $X(t)$

$$m_p(t) = \mathrm{E}[X^p(t)] = \int_{-\infty}^{\infty} x^p f_{X(t)}(x)dx$$

of *Random Sequence* X_n

$$m_p[n] = \mathrm{E}[X_n^p] = \sum_{x \in S_X} x^p P_{X_n}(x)$$

Autocovariance of a Process

- We could calculate the variance of $X(t)$, but the **covariance** function of a stochastic process provides very important information about the time structure of the process, and from the covariance we can always calculate the variance.
- Recall that $\mathbf{Cov}[X, Y]$ is an indication of how much information random variable X provides about random variable Y.
  - When the magnitude of the covariance is *high*, an observation of X provides an accurate indication of the value of Y.

Autocovariance of a Process

- If the two random variables are observations of $X(t)$ taken at two different times, t_1 seconds and $t_2 = t_1 + \tau$ seconds, the **covariance** indicates how much the process is likely to change in the τ seconds elapsed between t_1 and t_2.
- A *high* covariance indicates that the sample function is *unlikely to change* much in the τ-second interval.
- A covariance *near zero* suggests *rapid change*.
- This information is conveyed by the **autocovariance** function.

Autocovariance of a Process

The **autocovariance** function of the *stochastic process* $X(t)$ is

$$C_X(t, \tau) = \mathrm{Cov}\left[X(t), X(t+\tau)\right]$$

The **autocovariance** function of the *random sequence* Xn is

$$C_X[m, k] = \mathrm{Cov}[X_m, X_{m+k}]$$

At $\tau = k = 0$ implies $C_X(t, t) = \mathrm{Var}[X(t)]$ or $C_X[m, m] = Var[X_m]$.

Note in previous years this was written as: $C_{XX}(t, t+\tau)$, or $C_{XX}[m, m+k]$

(There is also a cross-covariance function that describes the relationship between two different random processes.)

Autocorrelation of a Process

The **autocorrelation** function of a stochastic process is closely related to the **autocovariance** function.

The **autocorrelation** function of the stochastic process $X(t)$ is

$$R_X(t, \tau) = \mathrm{E}[X(t)X(t+\tau)]$$

or the random sequence X_n is

$$R_X[m, k] = \mathrm{E}[X_m X_{m+k}]$$

Autocovariance and Autocorrelation

The autocorrelation and autocovariance functions of the process $X(t)$ satisfy

$$C_X(t,\tau) = R_X(t,\tau) - \mu_X(t)\mu_X(t+\tau)$$

or the random sequence Xn satisfy

$$C_X[n,k] = R_X[n,k] - \mu_X(n)\mu_X(n+k)$$

Example 13.14

Find the autocovariance $C_X(t\,\tau)$ and autocorrelation $R_X(t,\tau)$ of the Brownian motion process $X(t)$.