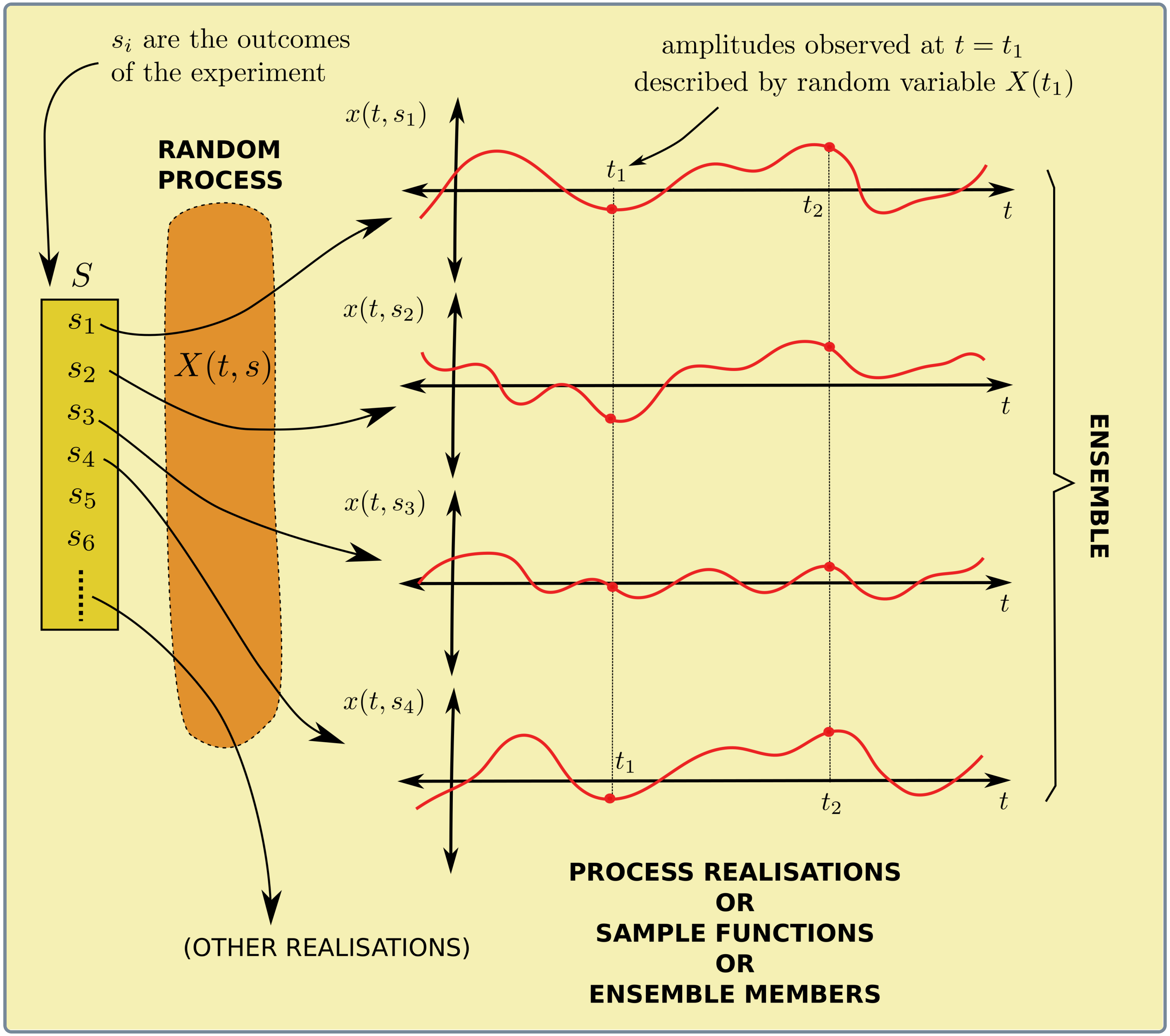
Systems and Signals 344

2025

Lecture 29

The Poisson Process

Sections 13.4-13.5



s_i are the outcomes of the experiment
amplitudes observed at $t = t_1$ described by random variable $X(t_1)$
RANDOM PROCESS
S
s_1
s_2
s_3
s_4
s_5
s_6
$X(t, s)$
$x(t, s_1)$
$x(t, s_2)$
$x(t, s_3)$
$x(t, s_4)$
t_1
t_2
t
ENSEMBLE
(OTHER REALISATIONS)
PROCESS REALISATIONS
OR
SAMPLE FUNCTIONS
OR
ENSEMBLE MEMBERS

Example 13.8

The Poisson Process

*The **Poisson process** is a memoryless counting process in which an arrival at a particular instant is independent of an arrival at any other instant.*

The Poisson Process

The Poisson process is one of the most widely-used counting processes. It is usually used in scenarios where we are counting the *occurrences* of certain events that appear to happen at a *certain rate*, but completely at random (without a certain structure).

The Poisson Process

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For example, suppose that from historical data, we know that earthquakes occur in a certain area with a rate of 22 per month. Other than this information, the timings of earthquakes seem to be completely random. Thus, we conclude that the Poisson process might be a good model for earthquakes.

The Poisson Process

In practice, the Poisson process or its extensions have been used to model:

- the number of car accidents at a site or in an area;
- the location of users in a wireless network;
  - Using Spatial Poisson Process
- the requests for individual documents on a web server;
- the outbreak of wars;
- photons landing on a photodiode.

Recap: The Poisson Process

Recall that if we know the average number of event arrivals expected in a specific interval of time, the **Poisson distribution** tells us the probability of witnessing exactly x arrivals in that specific interval.

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

We previously defined $\alpha = \lambda T$ (note that α is unit-less)

Counting Process

- A **counting process** $N(t)$ starts at time $\mathbf{0}$ and counts the occurrences of events.
  - These events are generally called **arrivals** because counting processes are most often used to model something like the arrivals of customers at a service facility.
- Since we start at time $t = 0, \; n(t, s) = 0$ for all $t \leq 0$. Also, the number of arrivals up to any $t > 0$ is an integer that cannot decrease with time.

Therefore, formally we provide the following definition:

A stochastic process $N(t)$ is a **counting process** if for every sample function, $n(t, s) = 0$ for $t < 0$ and $n(t, s)$ is *integer-valued* and *nondecreasing* with time.

The Poisson Process

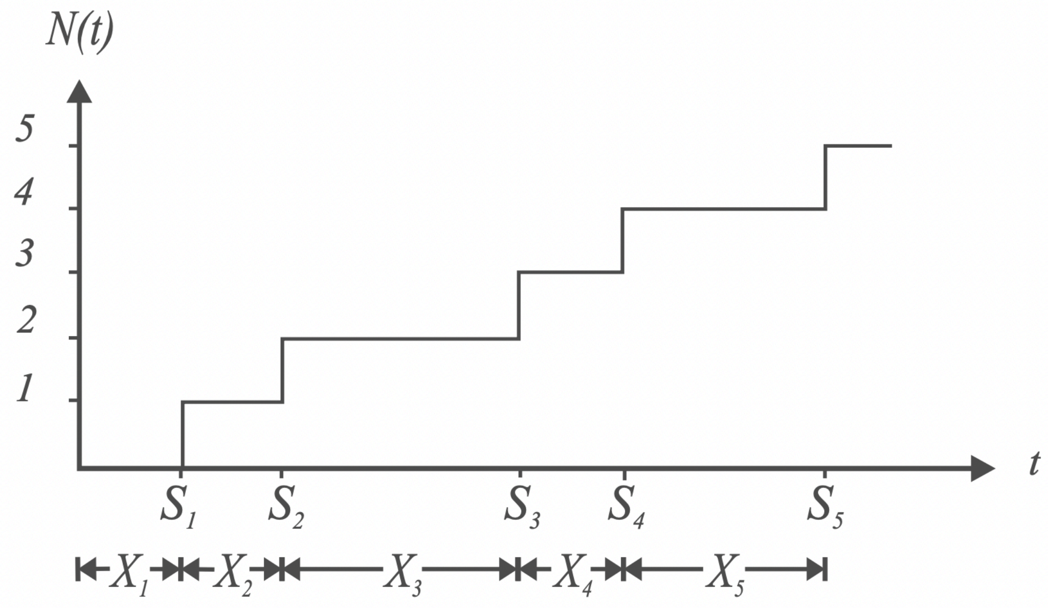


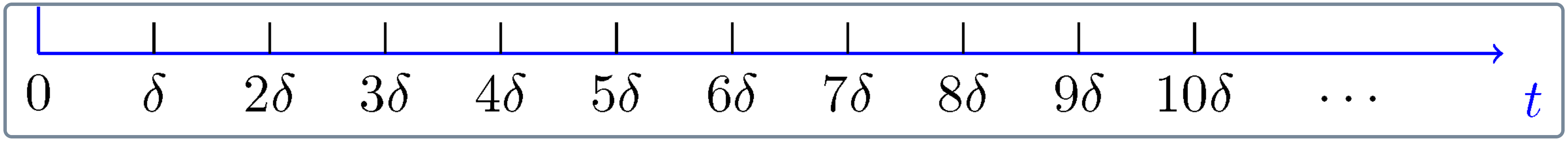
Figure 13.4 Sample path of a counting process.

The jumps in the sample function of a counting process mark the arrivals, and the number of arrivals in the interval $(t_0, t_1]$ is just $N(t_1) - N(t_0)$.

The Poisson Process

We can use a **Bernoulli process** $X_1, X_2, \ldots$ to derive a simple counting processing. We will then use this to describe the Poisson Process.

We now divide the half-line $[0, \infty)$ to tiny subintervals of length δ.



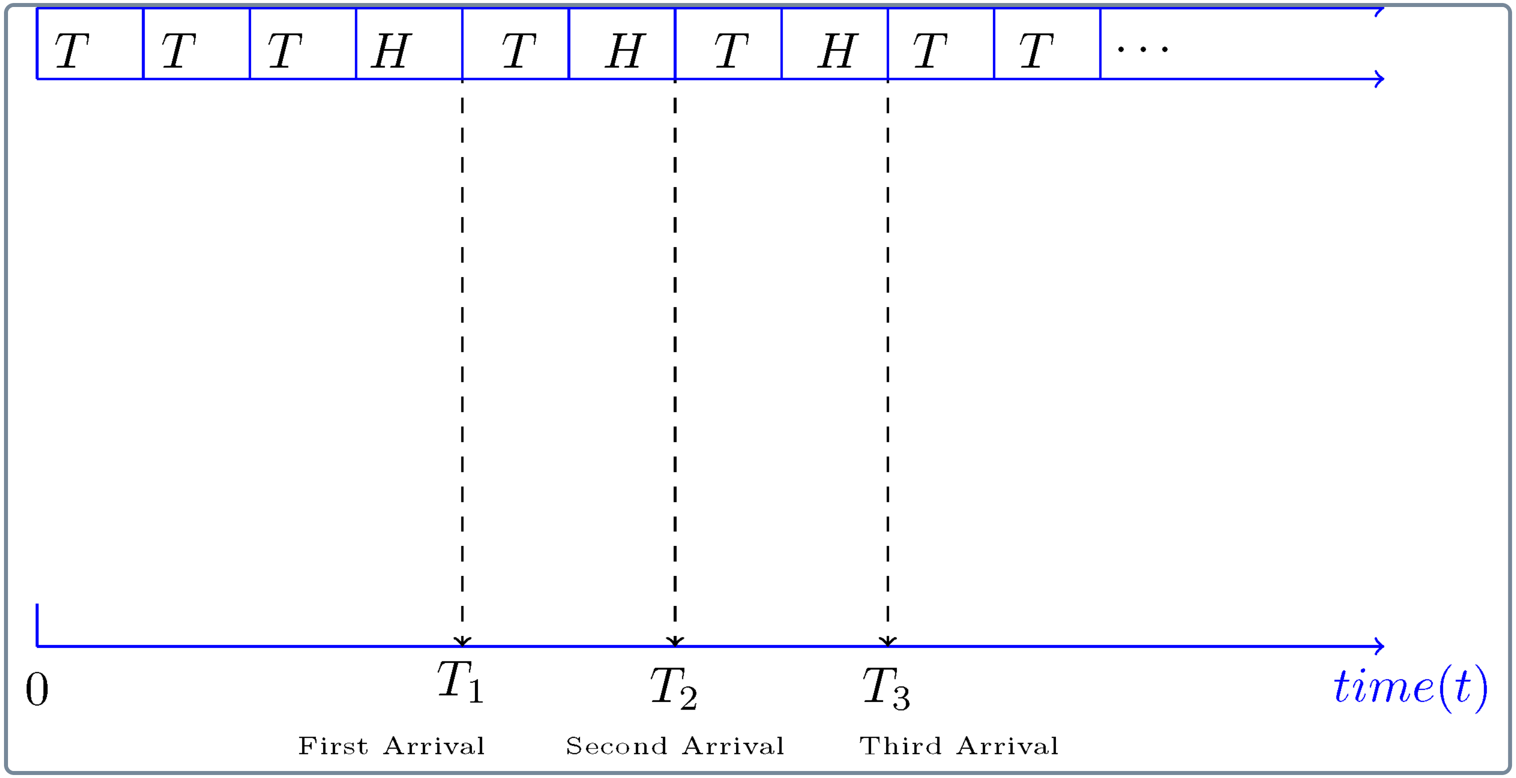
Each subinterval corresponds to a time slot of length δ. Thus, the intervals are $(0, \delta]$, $(\delta, 2\delta]$, $(2\delta, 3\delta]$, $\ldots$. More generally, the i-th interval is $((i-1)\delta, i\delta]$.

We assume that in each time slot, we, for example, toss a coin for which

$$P(X_i = 1) = P(\text{Success}) = P(\text{Heads}) = p = \lambda\delta.$$

The Poisson Process

If the coin lands heads up, we say that we have an arrival in that subinterval. Otherwise, we say that we have no arrival in that interval.



Here, we have an arrival at time $T = i\delta$, if the i-th coin flip results in a heads. (e.g. $T_1 = 4\delta, T_2 = 6\delta$)

The Poisson Process

Now, let N_m (a random sequence) be defined as the *number of arrivals* (number of heads) from time 0 to time T (i.e. $(0, T]$) with $T = m\delta$. Where m is the number of intervals (or number of coin flips).

We conclude that N_m is **Binomial**$(m, p = \lambda\delta = \frac{\lambda T}{m})$, with the following PMF:

$$P_{N_m}(n) = \binom{m}{n}\left(\frac{\lambda T}{m}\right)^n\left(1 - \frac{\lambda T}{m}\right)^{m-n}$$

Recall also that $\alpha = \lambda T$

The Poisson Process

Let N_m be **Binomial**$(m, p = \alpha/m)$. Let $\alpha > 0$ be a fixed real number. Then, the PMF of N_m converges to a **Poisson** (α) PMF, as $m \to \infty$.

That is, for any $n \in \{0, 1, 2, \ldots\}$, we have

$$\lim_{m\to\infty} P_{N_m}(n) = P_{N(T)}(n) = \frac{\alpha^n e^{-\alpha}}{n!}$$

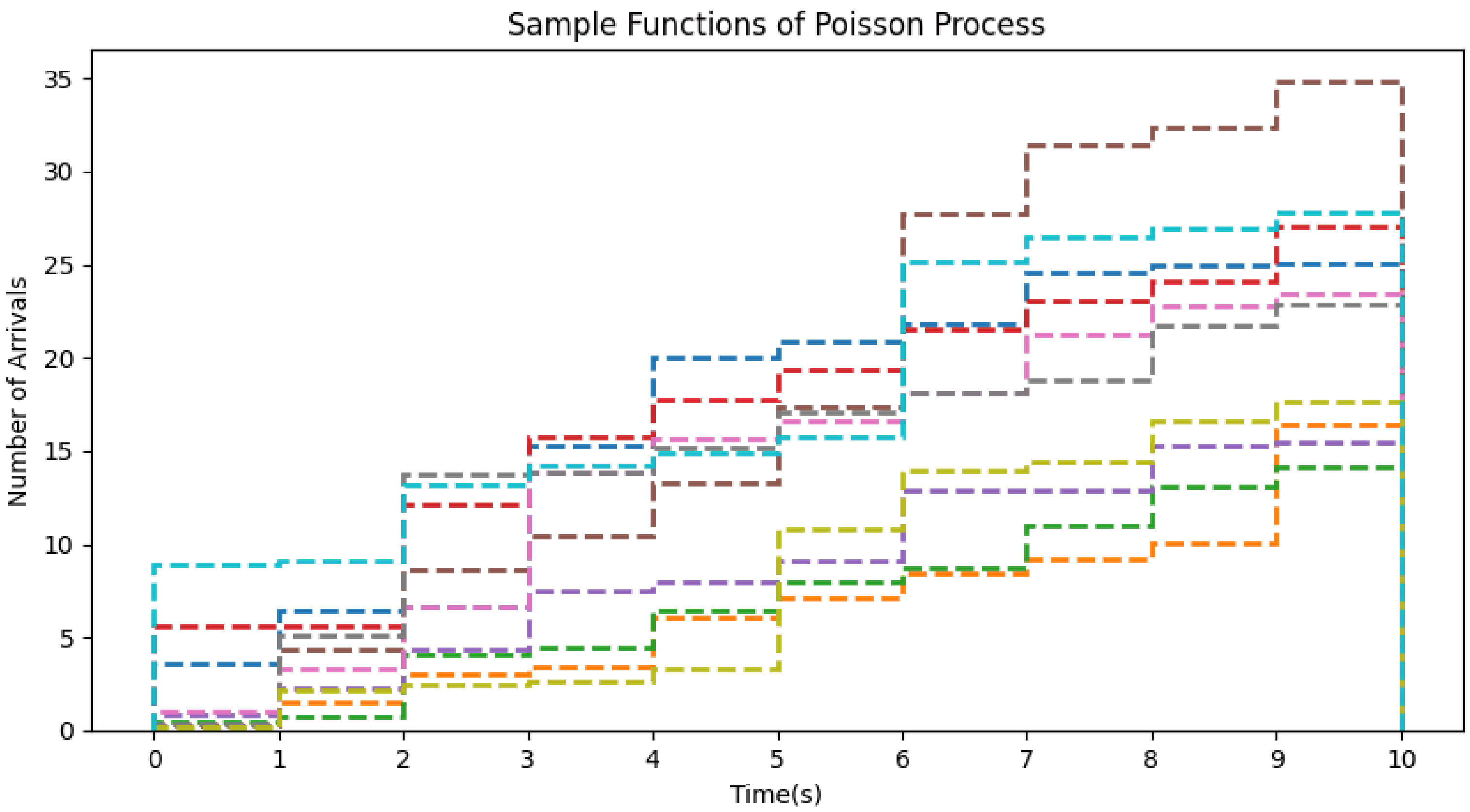
(This is Theorem 3.8 in the textbook)

The Poisson Process

- We can generalize this argument to say that for any interval $(t_0, t_1]$, the number of arrivals would have a Poisson PMF with parameter λT where $T = t_1 - t_0$.
  - The number of arrivals in $(t_0, t_1]$ depends on the independent Bernoulli trials corresponding to that interval.
- Thus the number of arrivals in *nonoverlapping* intervals will be *independent*.
- In the limit as $\delta \to 0$ (or $m \to \infty$), we have obtained a **counting process** in which the number of arrivals in *any* interval is a Poisson random variable independent of the arrivals in any other nonoverlapping interval.
- We call this limiting process a **Poisson process**.

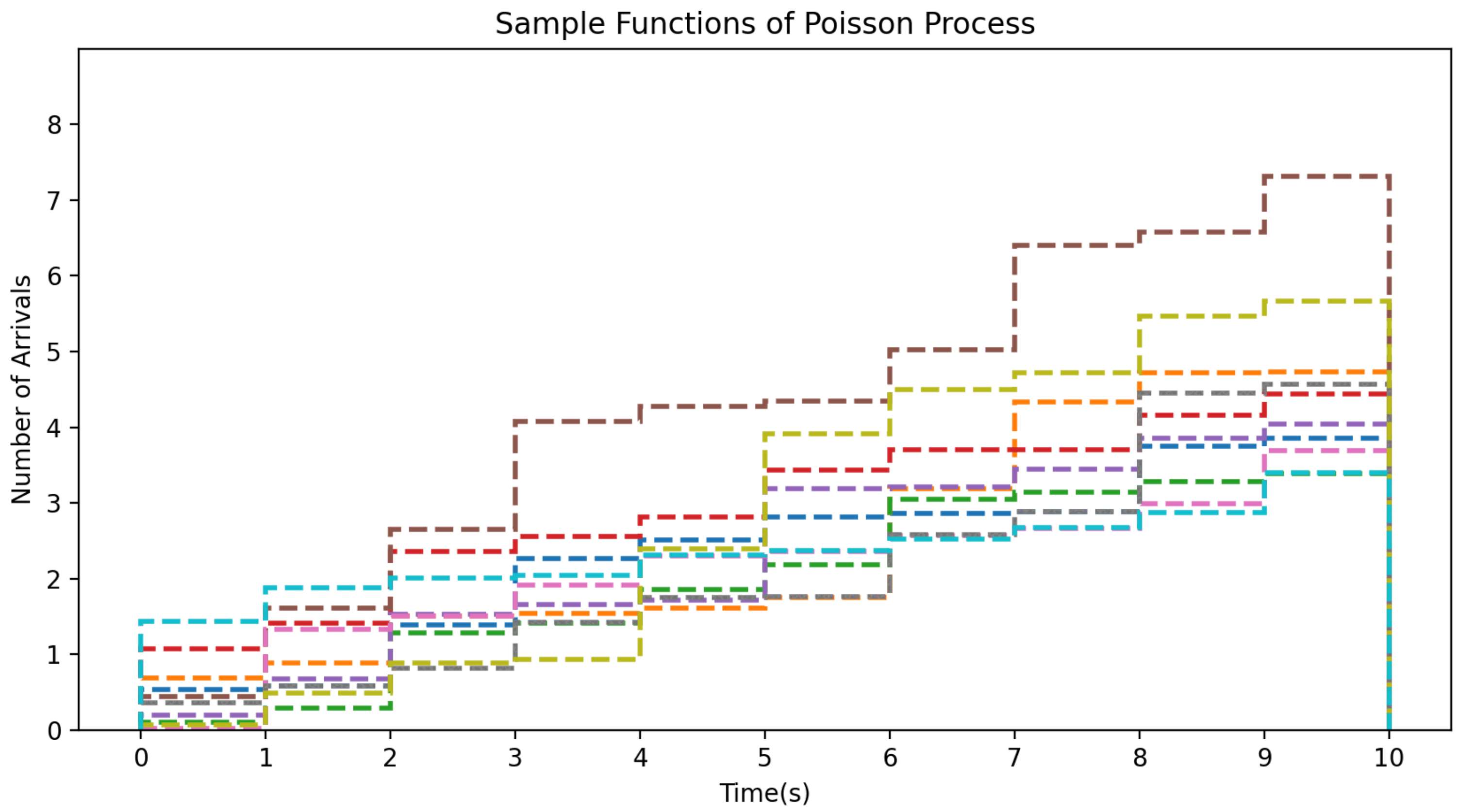
The Poisson Process

$\lambda = 2$ arrivals per second



The Poisson Process

$\lambda = 0.5$ arrivals per second



The Poisson Process

A counting process $N(t)$ is a **Poisson process** of rate λ if

1. The number of arrivals in any interval $(t_0, t_1]$, $N(t_1) - N(t_0)$, is a **Poisson random variable** with *expected value* $\lambda(t_1 - t_0)$.
2. For any pair of nonoverlapping intervals $(t_0, t_1]$ and $(t'_0, t'_1]$, the number of arrivals in each interval, $N(t_1) - N(t_0)$ and $N(t'_1) - N(t'_0)$, respectively, are *independent* random variables.

We call λ the *rate* of the process because the expected number of arrivals per unit time is $\dfrac{E[N(t)]}{T} = \lambda$.

The Poisson Process

By the definition of a Poisson random variable, $M = N(t_1) - N(t_0)$ has the PMF (with $\alpha = \lambda(t_1 - t_0)$)

$$P_M(m) = \begin{cases} \frac{[\lambda(t_1 - t_0)]^m e^{-\lambda(t_1 - t_0)}}{m!} & m = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The Poisson Process

For a set of time instants $t_1 < t_2 < \cdots < t_k$, we can use the property that the number of arrivals in *nonoverlapping* intervals are independent to write the joint PMF of $N(t_1), \ldots, N(t_k)$ as a product of probabilities.

For a Poisson process $N(t)$ of rate λ, the joint PMF of $\mathbf{N} = [N(t_1), \ldots, N(t_k)]$ for ordered time instances $t_1 < t_2 < \cdots < t_k$

$$P_{\mathbf{N}}(\mathbf{n}) = \begin{cases} \frac{\alpha_1^{n_1} e^{-\alpha_1}}{n_1!} \frac{\alpha_2^{n_2-n_1} e^{-\alpha_2}}{(n_2-n_1)!} \cdots \frac{\alpha_k^{n_k-n_{k-1}} e^{-\alpha_k}}{(n_k-n_{k-1})!} & 0 \le n_1 \le \cdots \le n_k, \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha_1 = \lambda t_1$, and for $i = 2, \ldots, k$, $\alpha_i = \lambda(t_i - t_{i-1})$.

The Poisson Process

- Keep in mind that the independent intervals property of the Poisson process must hold for *very small intervals*. For example, the number of arrivals in $(t, t+\delta]$ must be independent of the arrival process over $[0, t]$ no matter how small we choose $\delta > 0$.
- Essentially, the probability of an arrival during any instant is independent of the past history of the process.
    - In this sense, the Poisson process is **memoryless**.
- This memoryless property can also be seen when we examine the times between arrivals. The random time X_n between arrival $n-1$ and arrival n is called the n^{th} **interarrival time**.
- (In addition, we call the time X_1 of the first arrival the first interarrival time even though there is no previous arrival.)

The Poisson Process

For a Poisson process of rate λ, the *interarrival times* $X_1, X_2, \ldots$ are an iid random sequence with the **exponential** PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Recall that the **exponential distribution** is used to model the waiting time before a certain event occurs (where x is the time).

The inverse is also true:

A counting process ($N(t)$) with independent exponential (λ) interarrivals $X_1, X_2, \ldots$ is a Poisson process of rate λ.

Properties of the Poisson Process

The sum $N(t) = N_1(t) + N_2(t)$ of independent Poisson processes $N_1(t)$ and $N_2(t)$ is a Poisson process.

The Poisson process $N(t)$ can be decomposed into two independent Poisson processes $N_1(t)$ and $N_2(t)$.

- The **memoryless property** of the Poisson process can also be seen in the exponential interarrival times.
  - Since $P(X_n > x) = e^{-\lambda x}$, then
    $$P(X_n > t + x | X_n > t) = \frac{P(X_n > t + x, X_n > t)}{P(X_n > t)} = e^{-\lambda x}$$
- This means that if the arrival has not occurred by time t, the additional time until the arrival, $X_n - t$, has the same exponential distribution as X_n.
- That is, no matter how long we have waited for the arrival, the remaining time until the arrival is still an exponential (λ) RV.
- The consequence is that if we start to watch a Poisson process *at any time* t, we see a stochastic process that is indistinguishable from a Poisson process started at time 0.
- This interpretation is the basis for ways of **composing** and **decomposing** Poisson processes.

Properties of the Poisson Process

- This interpretation is the basis for ways of **composing** and **decomposing** Poisson processes.
- First we consider the sum $N(t) = N_1(t) + N_2(t)$ of two *independent* Poisson processes $N_1(t)$ and $N_2(t)$.
- Clearly, $N(t)$ is a counting process since any sample function of $N(t)$ is nondecreasing.
- Since interarrival times of both $N_1(t)$ and $N_2(t)$ are *continuous* exponential random variables, the probability that both processes have arrivals at the exact same time is zero.
    - Thus $N(t)$ increases by one arrival at a time.
- Also, in Chapter 9 (Theorem 9.7) it is showed that the sum of independent Poisson random variables is also Poisson.

Properties of the Poisson Process

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes of rates λ_1 and λ_2. The counting process $N(t) = N_1(t) + N_2(t)$ is a Poisson process of rate $\lambda_1 + \lambda_2$.

This describes the **composition** of a Poisson process.

Properties of the Poisson Process

Example 13.12

Cars, trucks, and buses arrive at a toll booth as independent Poisson processes with rates $\lambda_c = 1.2$ cars/minute, $\lambda_t = 0.9$ trucks/minute, and $\lambda_b = 0.7$ buses/minute. In a 10-minute interval, what is the PMF of N, the number of vehicles (cars, trucks, or buses) that arrive?

By Theorem 13.5, the arrival of vehicles is a Poisson process of rate $\lambda = 1.2+0.9+0.7 = 2.8$ vehicles per minute. In a 10-minute interval, $\lambda T = 28$ and N has PMF

$$P_N(n) = \begin{cases} 28^n e^{-28}/n! & n = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{13.21}$$

Properties of the Poisson Process

We can also examine the **decomposition** of a Poisson process into two separate processes.

Properties of the Poisson Process

- Suppose whenever a Poisson process $N(t)$ has an arrival, we flip a *biased* coin to decide whether to call this a $\textbf{type 1}$ or $\textbf{type 2}$ arrival.
- That is, each arrival of $N(t)$ is independently labeled either $\textbf{type 1}$ with probability p or $\textbf{type 2}$ with probability $1 - p$.
- This results in two counting processes, $N_1(t)$ and $N_2(t)$, where $N_i(t)$ denotes the number of $\textbf{type}\ i$ arrivals by time t.
- We will call this procedure of breaking down the $N(t)$ processes into two counting processes a **Bernoulli decomposition**.

The counting processes $N_1(t)$ and $N_2(t)$ derived from a **Bernoulli decomposition** of the Poisson process $N(t)$ are independent Poisson processes with rates λp and $\lambda(1 - p)$.

Properties of the Poisson Process

The last few statements leads to the following conclusion:

Let $N(t) = N_1(t) + N_2(t)$ be the sum of two independent Poisson processes with rates λ_1 and λ_2. Given that the $N(t)$ process has an arrival, the conditional probability that the arrival is from $N_1(t)$ is

$$p = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Properties of the Poisson Process

Example 13.13

A corporate Web server records hits (requests for HTML documents) as a Poisson process at a rate of 10 hits per second. Each page is either an internal request (with probability 0.7) from the corporate intranet or an external request (with probability 0.3) from the Internet. Over a 10-minute interval, what is the joint PMF of I, the number of internal requests, and X, the number of external requests?

By Theorem 13.6, the internal and external request arrivals are independent Poisson processes with rates of 7 and 3 hits per second. In a 10-minute (600-second) interval, I and X are independent Poisson random variables with parameters $\alpha_I = 7(600) = 4200$ and $\alpha_X = 3(600) = 1800$ hits. The joint PMF of I and X is

$$
\begin{aligned}
P_{I,X}(i,x) &= P_I(i)\, P_X(x) \\
&= \begin{cases} \frac{(4200)^i e^{-4200}}{i!} \frac{(1800)^x e^{-1800}}{x!} & i, x \in \{0, 1, \ldots\}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\tag{13.24}
$$