Systems and Signals 344

2025

Lecture 28

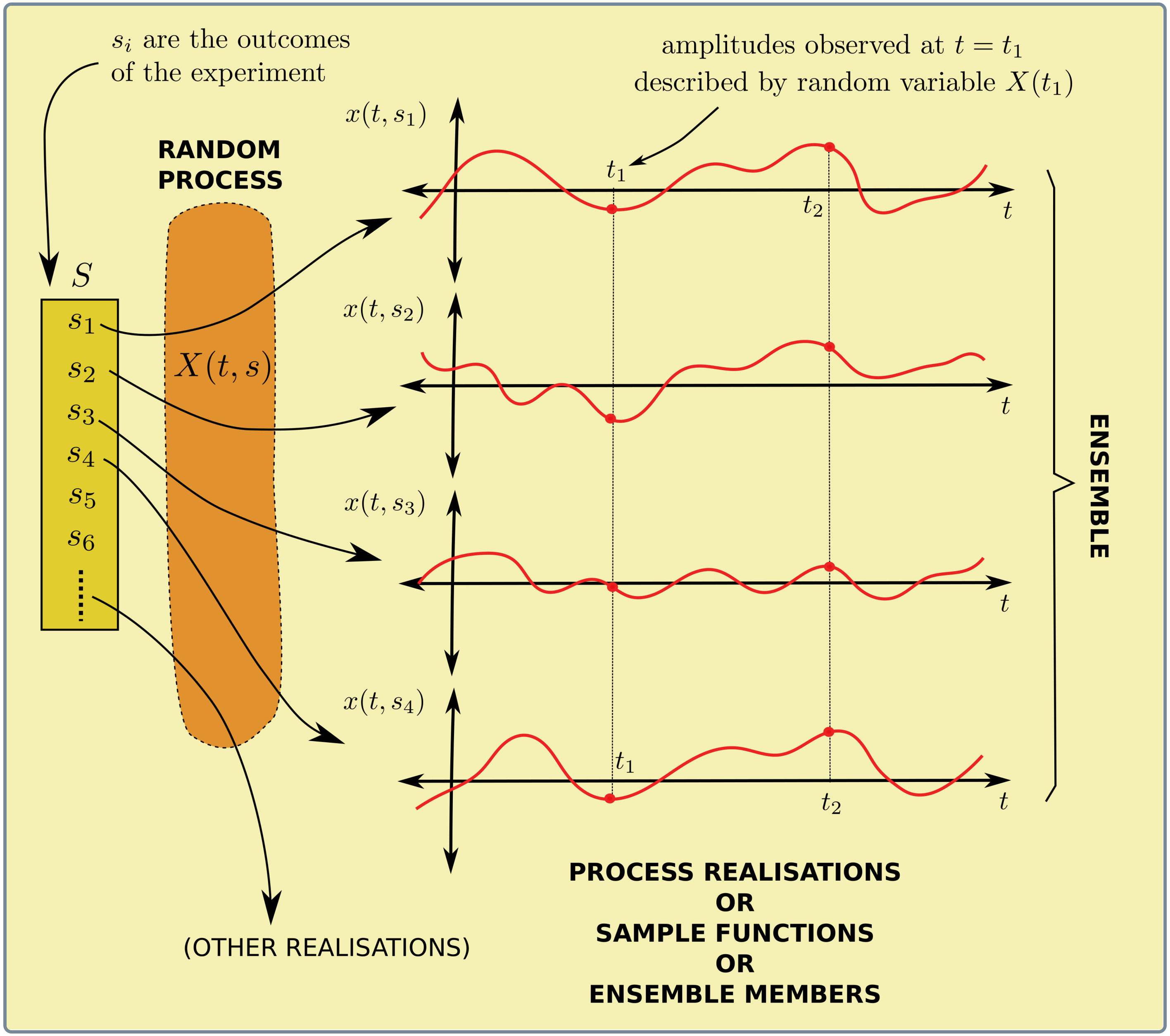
Stochastic Processes

Sections 13.1-13.3

Stochastic Processes

- Our study of probability refers to an experiment consisting of a procedure and observations.
- When we study **random variables**, each *observation* corresponds to a *number*.
- When we study **stochastic processes**, each *observation* corresponds to a *function of time*.
    - "Stochastic" means randomness that can be *modeled* with a *probability distribution*
    - "Process" in this context means function of time.
    - Therefore, when we study stochastic processes, we study random functions of time.

The terms **random process** and **stochastic process** are considered synonyms and are used interchangeably.


s_i are the outcomes of the experiment
amplitudes observed at $t = t_1$ described by random variable $X(t_1)$
RANDOM PROCESS
S
s_1
s_2
s_3
s_4
s_5
s_6
$X(t, s)$
$x(t, s_1)$
$x(t, s_2)$
$x(t, s_3)$
$x(t, s_4)$
t_1
t_2
t
ENSEMBLE
(OTHER REALISATIONS)
PROCESS REALISATIONS
OR
SAMPLE FUNCTIONS
OR
ENSEMBLE MEMBERS

Stochastic Processes

*A **stochastic process** $X(t)$ consists of an experiment with a probability measure $P(\cdot)$ defined on a sample space S and a mapping that assigns a time function $x(t, s)$ (called the **sample function**) to each outcome s in the sample space of the experiment.*

- $X(t)$ is both the name of the process and the name of the *random variable* observed at time t. (note by fixing t it is just a random variable)
- The stochastic process $X(t)$ is a "family" or **ensemble** of time functions $x(t, s)$ that can result from an experiment.
- Once s is known, the resulting function is deterministic.

Stochastic Processes

- Just as with random variables, one of the main benefits of the stochastic process model is that it lends itself to calculating *averages*.
- Corresponding to the two-dimensional nature of a stochastic process, there are two kinds of averages.
  1. With t fixed at $t = t_0$, $X(t_0)$ is a random variable, and we have the averages (for example, the expected value and the variance) that we have studied already. They are referred to as **ensemble averages**.
  2. The other type of average applies to a specific sample function, $x(t, s_0)$, and produces a typical number for this sample function. This is a **time average of the sample function**.

Stochastic Processes

Example 13.1

Starting at launch time $t = 0$, let $X(t)$ denote the temperature in Kelvins on the surface of a space shuttle. With each launch s, we record a temperature sequence $x(t, s)$. The ensemble of the experiment can be viewed as a catalog of the possible temperature sequences that we may record. For example,

$$x(8073.68, 175) = 207 \tag{13.1}$$

indicates that in the 175th entry in the catalog of possible temperature sequences, the temperature at $t = 8073.68$ seconds after the launch is 207 K.

Stochastic Processes

Example 13.2

In Example 13.1 of the space shuttle, over all possible launches, the average temperature after 8073.68 seconds is $\mathrm{E}[X(8073.68)] = 217$ K. This is an ensemble average taken over all possible temperature sequences. In the 175th entry in the catalog of possible temperature sequences, the average temperature over that space shuttle mission is

$$\frac{1}{671,208.3} \int_0^{671,208.3} x(t, 175)\, dt = 187.43 \text{ K}, \tag{13.2}$$

where the integral limit $671,208.3$ is the duration in seconds of the shuttle mission.

Example

Temperatures

Stochastic Processes

- Just as we developed different ways of analyzing **discrete** and **continuous** random variables, we can define categories of stochastic processes that can be analyzed using different mathematical techniques.
- To establish these categories, we characterize both the range of possible values at any instant t as well as the time instants at which changes in the random process can occur.
  - (both of the magnitude axis and time axis could be continuous or discrete)

Stochastic Processes

Discrete-Value and Continuous-Value Processes

$X(t)$ is a *discrete-value process* if the set of all possible values of $X(t)$ at all times t is a countable set S_X; otherwise $X(t)$ is a *continuous-value process*.

Discrete-Time and Continuous-Time Processes

The stochastic process $X(t)$ is a *discrete-time process* if $X(t)$ is defined only for a set of time instants, $t_n = nT$, where T is a constant and n is an integer; otherwise $X(t)$ is a *continuous-time process*.

A *discrete-time process* is also called a **Random Sequence** A random sequence X_n is an ordered sequence of random variables

$$X_0, X_1, \ldots$$

Stochastic Processes

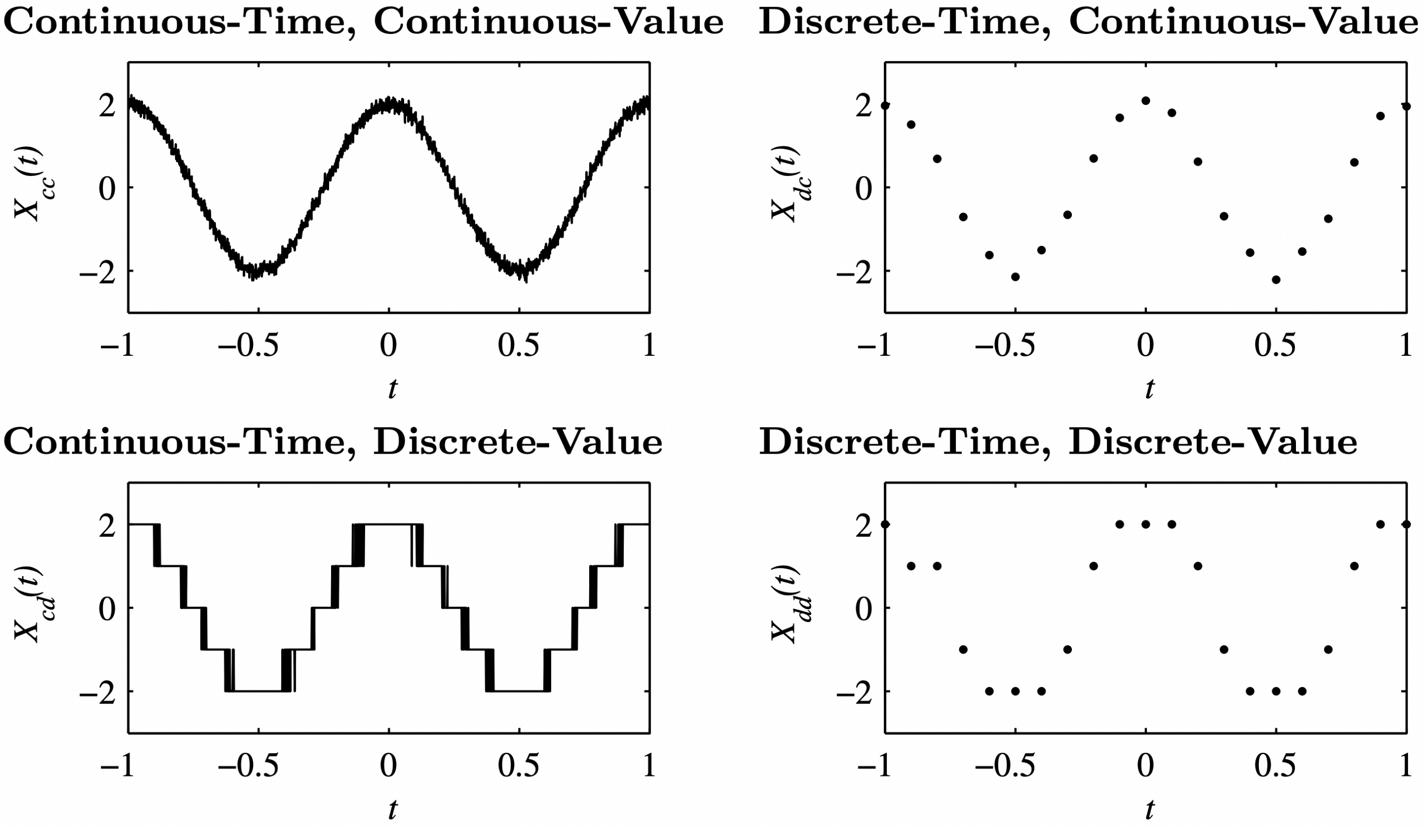


Figure 13.3 Sample functions of four kinds of stochastic processes. $X_{cc}(t)$ is a continuous-time, continuous-value process. $X_{dc}(t)$ is discrete-time, continuous-value process obtained by sampling X_{cc}) every 0.1 seconds. Rounding $X_{cc}(t)$ to the nearest integer yields $X_{cd}(t)$, a continuous-time, discrete-value process. Lastly, $X_{dd}(t)$, a discrete-time, discrete-value process, can be obtained either by sampling $X_{cd}(t)$ or by rounding $X_{dc}(t)$.

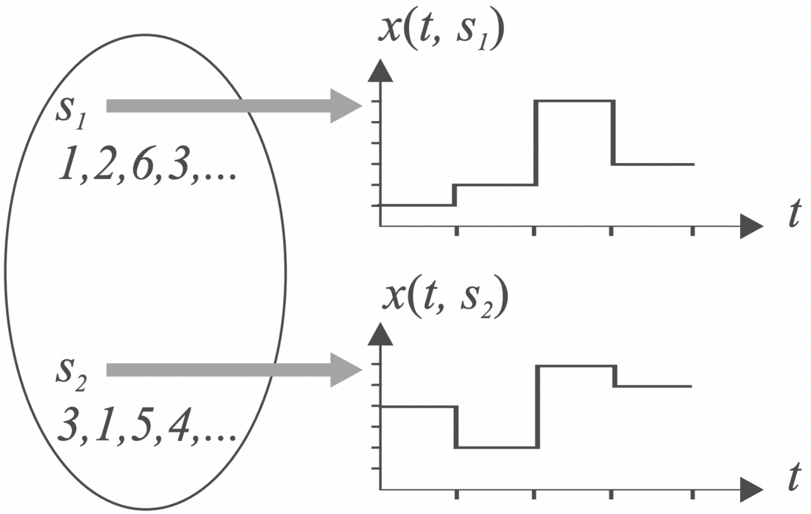
Stochastic Processes

- Although the stochastic process definition refers to one experiment producing an observation $\boldsymbol{s}$, associated with a sample function $x(t, \boldsymbol{s})$, our experience with practical applications of stochastic processes can better be described in terms of an *ongoing sequence of observations of random events*.

Stochastic Processes

Example 13.5

Suppose that at time instants $T = 0, 1, 2, \ldots$, we roll a die and record the outcome N_T where $1 \leq N_T \leq 6$. We then define the random process $X(t)$ such that for $T \leq t < T+1$, $X(t) = N_T$. In this case, the experiment consists of an infinite sequence of rolls and a sample function is just the waveform corresponding to the particular sequence of rolls. This mapping is depicted on the right.



Example 13.4

Consider an experiment in which we record $M(t)$, the number of active calls at a telephone switch at time t, at each second over an interval of 15 minutes. One trial of the experiment might yield the sample function $m(t,s)$ shown in Figure 13.2. Each time we perform the experiment, we would observe some other function $m(t,s)$. The exact $m(t,s)$ that we do observe will depend on many random variables including the number of calls at the start of the observation period, the arrival times of the new calls, and the duration of each call. An ensemble average is the average number of calls in progress at $t = 403$ seconds. A time average is the average number of calls in progress during a specific 15-minute interval.

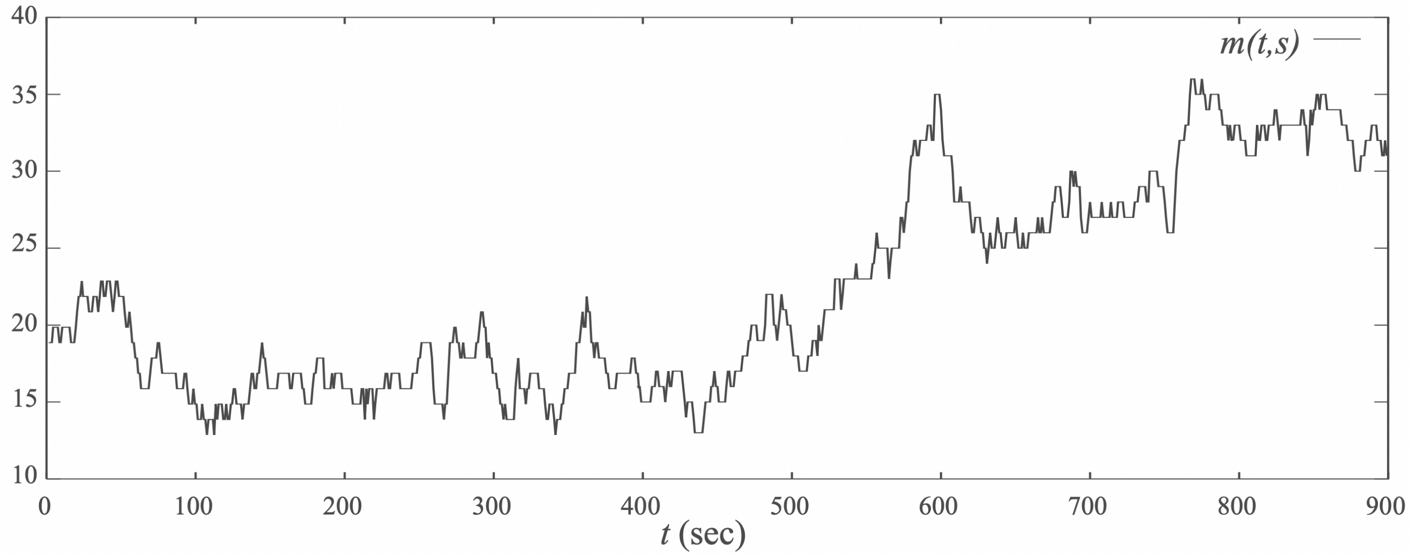


Figure 13.2 A sample function $m(t,s)$ of the random process $M(t)$ described in Example 13.4.

In Example 13.4, the sample function $m(t, s)$ is the result of a sequence of experiments, with a new experiment performed every second. The observations of each experiment produce several random variables related to the sample functions of the stochastic process, together generating the model of the entire process $M(t)$.
For example, the observations related to the waveform $m(t, s)$.

- $m(0, s)$, the number of ongoing calls at the start of the experiment,
- $X_1, \ldots, X_{m(0,s)}$, the remaining time in seconds of each of the $m(0, s)$ ongoing calls,
- N, the number of new calls that arrive during the experiment,
- $S_1, \ldots, S_N$, the arrival times in seconds of the N new calls,
- $Y_1, \ldots, Y_N$, the call durations in seconds of each of the N new calls.

Random Variables from Random Processes

- Suppose we observe a stochastic process at a particular time instant t_1.
- In this case, each time we perform the experiment, we observe a sample function $x(t, s)$ and that sample function specifies the value of $x(t_1, s)$.
- Each time we perform the experiment, we have a new s and we observe a new $x(t_1, s)$.
- Therefore, each $x(t_1, s)$ is a *sample value* of a *random variable*. We use the notation $X(t_1)$ for this random variable.
- Like any other random variable, it has either a PDF $f_{X(t_1)}(x)$ or a PMF $P_{X(t_1)}(x)$.

Random Variables from Stochastic Processes

- If we sample a stochastic process (continuous or discrete) $X(t)$ at k time instants $t_1, \ldots, t_k$ (for any value of k), we obtain the k-dimensional random vector $\mathbf{X} = [X(t_1) \quad \cdots \quad X(t_k)]'$.
  - Remember a random vector is simply a *joint probability distribution*.
- This random vector is described by the joint PMF $P_{\mathbf{X}}(x)$ for a *discrete-value process* $X(t)$ or by the joint PDF $f_{\mathbf{X}}(x)$ for a *continuous-value process* $X(t)$.
- That means that knowledge of the joint PDF $f_{X(t1),\ldots,X(t_k)}(x_1, \ldots, x_k)$ for all k will allow us to describe a random process without reference to an underlying experiment.

Random Variables from Stochastic Processes

- The same rules apply to these $X(t_i)$ random variables as to any random variable.
  - PMF/PDF to CDF conversion (and vice versa)
  - Marginalisation
  - Independence
  - Expected values
  - Moments

IID Random Sequences

- An **independent identically distributed (iid) random sequence** is a random sequence X_n in which $\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$ are iid random variables.
- An iid random sequence occurs whenever we perform independent trials of an experiment at a constant rate.
- An iid random sequence can be either *discrete-value* or *continuous-value*, but it is per definition *discrete-time*.
  - Therefore, either $P_{X_i}(x) = P_X(x)$ or $f_{X_i}(x) = f_X(x)$.

IID Random Sequences

Let X_n denote an *iid random sequence* with sample vector

$$\mathbf{X} = [X_{n_1} \quad \cdots \quad X_{nk}]'.$$

Discrete-value process

$$P_{\mathbf{X}}(\mathbf{x}) = P_X(x_1)P_X(x_2)\cdots P_X(x_k) = \prod_{i=1}^{k} P_X(x_i)$$

Continuous-value process

$$f_{\mathbf{X}}(\mathbf{x}) = f_X(x_1)f_X(x_2)\cdots f_X(x_k) = \prod_{i=1}^{k} f_X(x_i)$$

IID Random Sequences

Of all iid random sequences, perhaps the Bernoulli random sequence is the simplest.

A **Bernoulli (p) process** X_n is an iid random sequence in which each X_i is a **Bernoulli (p) random variable**. In other words, it either produces a "success" or a "failure", usually indicated with either 1 or 0.

Example 13.11

For a Bernoulli (p) process X_n, find the joint PMF of $\mathbf{X} = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}'$.