Systems and Signals 344

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Lecture 22

Random Vectors: Expected Value Vector and Correlation Matrix, Gaussian Random Vectors

Sections 8.4-8.5

Recap: Vector Notation

A **random vector** is a column vector $\mathbf{X} = [X_1 \quad \dots \quad X_n]'$.
Each X_i is a random variable.

A **sample value of a random vector** is a column vector
$$\mathbf{x} = [x_1 \quad \dots \quad x_n]'.$$
The i^{th} component, x_i, of the vector $\mathbf{x}$ is a sample value of a random variable, X_i.

We use similar notation for a function $g(\mathbf{X}) = g(X_1, \dots, X_n)$ of n random variables and a function $g(\mathbf{x}) = g(x_1, \dots, x_n)$ of n numbers.

Recap: Vector Notation

The **CDF** of a random vector $\mathbf{X}$ is

$$F_{\mathbf{X}}(\mathbf{x}) = F_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

The **PMF** of a *discrete* random vector $\mathbf{X}$ is

$$P_{\mathbf{X}}(\mathbf{x}) = P_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

The **PDF** of a *continuous* random vector $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

Expected Value Vector and Correlation Matrix

The ***expected value of a random vector*** *is a vector containing the expected values of the components of the vector.*

The ***covariance of a random vector*** *is a symmetric matrix containing the variances of the components of the random vector and the covariances of all pairs of random variables in the random vector.*

Expected Value Vector

The **expected value of a random vector $\mathbf{X}$** is a column vector

$$\mathrm{E}[\mathbf{X}] = \boldsymbol{\mu}_{\mathbf{X}} = [\mathrm{E}[X_1] \;\; \mathrm{E}[X_2] \;\; \dots \;\; \mathrm{E}[X_n]]'$$

In general, it can be said that expectation applies *element-wise* to vectors and matrices.

Expected Value Vector

In general, it can be said that expectation applies *element-wise* to vectors and matrices.

$$\text{If } \mathbf{A} = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{m1} & \cdots & X_{mn} \end{bmatrix}$$

$$\mathrm{E}[\mathbf{A}] = \begin{bmatrix} \mathrm{E}[X_{11}] & \cdots & \mathrm{E}[X_{1n}] \\ \vdots & \ddots & \vdots \\ \mathrm{E}[X_{m1}] & \cdots & \mathrm{E}[X_{mn}] \end{bmatrix}$$

Correlation Matrix

- The **correlation** and **covariance** are numbers that contain important information about *a pair* of random variables.
- Corresponding information about random vectors is reflected in the **set of correlations** and the **set of covariances** of *all pairs* of components.
    - These sets are referred to as **second-order statistics** (showing relationship between variables). They have a concise matrix notation.
- To establish the notation, we first observe that for random vectors $\mathbf{X}$ with n components and $\mathbf{Y}$ with m components, the set of all products, X_iY_j , is contained in the $n \times m$ random matrix $\mathbf{XY}'$
- If $\mathbf{Y} = \mathbf{X}$, the random matrix $\mathbf{XX}'$ contains all products, X_iX_j, of components of $\mathbf{X}$.

Correlation Matrix

An example of a vector containing three random variables

If $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$, what are the components of $\mathbf{X}\mathbf{X}'$?

$$\mathbf{X}\mathbf{X}' = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} X_1^2 & X_1X_2 & X_1X_3 \\ X_2X_1 & X_2^2 & X_2X_3 \\ X_3X_1 & X_3X_2 & X_3^2 \end{bmatrix}. \tag{8.22}$$

Vector Correlation

The **correlation of a random vector $\mathbf{X}$** is an $n \times n$ matrix $\mathbf{R}_\mathbf{X}$ with i, j^{th} element $R_X(i,j) = \mathrm{E}[X_i X_j]$.

In vector notation is called **autocorrelation**,

$$\mathbf{R}_\mathbf{X} = \mathrm{E}[\mathbf{X}\mathbf{X}']$$

For the example of the vector with three random variables:

If $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$, the correlation matrix of $\mathbf{X}$ is

$$\mathbf{R}_\mathbf{X} = \begin{bmatrix} \mathrm{E}\left[X_1^2\right] & \mathrm{E}\left[X_1X_2\right] & \mathrm{E}\left[X_1X_3\right] \\ \mathrm{E}\left[X_2X_1\right] & \mathrm{E}\left[X_2^2\right] & \mathrm{E}\left[X_2X_3\right] \\ \mathrm{E}\left[X_3X_1\right] & \mathrm{E}\left[X_3X_2\right] & \mathrm{E}\left[X_3^2\right] \end{bmatrix} = \begin{bmatrix} \mathrm{E}\left[X_1^2\right] & r_{X_1,X_2} & r_{X_1,X_3} \\ r_{X_2,X_1} & \mathrm{E}\left[X_2^2\right] & r_{X_2,X_3} \\ r_{X_3,X_1} & r_{X_3,X_2} & \mathrm{E}\left[X_3^2\right] \end{bmatrix}.$$

Vector Covariance

The **covariance of a random vector $\mathbf{X}$** is an $n \times n$ matrix $\mathbf{C_X}$ with components $C_X(i,j) = \text{Cov}[X_i, X_j]$.

In vector notation is called **autocovariance**,

$$\mathbf{C_X} = \mathbf{E}[(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})(\mathbf{X} - \boldsymbol{\mu}_\mathbf{X})']$$

For the example of the vector with three random variables (remember $\mathbf{Var}[X] = \sigma_X^2$):

If $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$, the covariance matrix of $\mathbf{X}$ is

$$\mathbf{C_X} = \begin{bmatrix} \text{Var}[X_1] & \text{Cov}[X_1, X_2] & \text{Cov}[X_1, X_3] \\ \text{Cov}[X_2, X_1] & \text{Var}[X_2] & \text{Cov}[X_2, X_3] \\ \text{Cov}[X_3, X_1] & \text{Cov}[X_3, X_2] & \text{Var}[X_3] \end{bmatrix} \tag{8.23}$$

Vector Covariance

For a random vector $\mathbf{X}$ with correlation matrix $\mathbf{R}_\mathbf{X}$, covariance matrix $\mathbf{C}_\mathbf{X}$, and vector expected value $\boldsymbol{\mu}_\mathbf{X}$,

$$\mathbf{C}_\mathbf{X} = \mathbf{R}_\mathbf{X} - \boldsymbol{\mu}_\mathbf{X}\boldsymbol{\mu}'_\mathbf{X}$$

Some texts prefer to write: $\mathbf{C}_{\mathbf{XX}}$ for $\mathbf{C}_\mathbf{X}$ and $\mathbf{R}_{\mathbf{XX}}$ for $\mathbf{R}_\mathbf{X}$, and these will always be *square matrices*.

Vector Cross-Correlation

The **cross-correlation of random vectors**, $\mathbf{X}$ with n components and $\mathbf{Y}$ with m components, is an $n \times m$ matrix $\mathbf{R_{XY}}$ with i, j^{th} element

$$R_{XY}(i,j) = \mathrm{E}[X_i Y_j].$$

In vector notation,

$$\boxed{\mathbf{R_{XY}} = \mathrm{E}[\mathbf{X}\mathbf{Y}']}$$

Vector Cross-Covariance

The **cross-covariance of a pair of random vectors**, $\mathbf{X}$ with n components and $\mathbf{Y}$ with m components, is an $n \times m$ matrix $\mathbf{C}_{\mathbf{XY}}$ with i, j^{th} element $C_{XY}(i, j) = \mathbf{E}[X_i Y_j]$.

In vector notation,

$$\mathbf{C}_{\mathbf{XY}} = \mathbf{E}[(\mathbf{X} - \boldsymbol{\mu}_{\mathbf{X}})(\mathbf{Y} - \boldsymbol{\mu}_{\mathbf{Y}})']$$

Example 8.7

Find the expected value $\mathrm{E}[\mathbf{X}]$, the correlation matrix $\mathbf{R_X}$, and the covariance matrix $\mathbf{C_X}$ of the two-dimensional random vector $\mathbf{X}$ with PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 2 & 0 \le x_1 \le x_2 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.26}$$

....

The elements of the expected value vector are

$$\mathrm{E}[X_i] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i f_{\mathbf{X}}(\mathbf{x})\, dx_1 dx_2 = \int_0^1 \int_0^{x_2} 2x_i\, dx_1 dx_2, \quad i = 1, 2. \tag{8.27}$$

The integrals are $\mathrm{E}[X_1] = 1/3$ and $\mathrm{E}[X_2] = 2/3$, so that $\boldsymbol{\mu}_{\mathbf{X}} = \mathrm{E}[\mathbf{X}] = \begin{bmatrix}1/3 & 2/3\end{bmatrix}'$. The elements of the correlation matrix are

$$\mathrm{E}\left[X_1^2\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 f_{\mathbf{X}}(\mathbf{x})\, dx_1 dx_2 = \int_0^1 \int_0^{x_2} 2x_1^2\, dx_1 dx_2, \tag{8.28}$$

$$\mathrm{E}\left[X_2^2\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2^2 f_{\mathbf{X}}(\mathbf{x})\, dx_1 dx_2 = \int_0^1 \int_0^{x_2} 2x_2^2\, dx_1 dx_2, \tag{8.29}$$

$$\mathrm{E}[X_1 X_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{\mathbf{X}}(\mathbf{x})\, dx_1 dx_2 = \int_0^1 \int_0^{x_2} 2x_1 x_2\, dx_1 dx_2. \tag{8.30}$$

These integrals are $\mathrm{E}[{X_1}^2] = 1/6$, $\mathrm{E}[{X_2}^2] = 1/2$, and $\mathrm{E}[X_1 X_2] = 1/4$.
Therefore,

$$\mathbf{R_X} = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}. \tag{8.31}$$

We use Theorem 8.7 to find the elements of the covariance matrix.

$$\mathbf{C_X} = \mathbf{R_X} - \boldsymbol{\mu}_{\mathbf{X}}\boldsymbol{\mu}_{\mathbf{X}}' = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix} = \begin{bmatrix} 1/18 & 1/36 \\ 1/36 & 1/18 \end{bmatrix}. \tag{8.32}$$

Linear Transform of a Random Vector

Consider the following definitions (X_i and Y_j are random variables, and a_{ij} and b_i are constants):

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Now consider the linear transform:

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$$

Linear Transform of a Random Vector

Mean

$$\boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{b}$$

Autocorrelation

$$\mathbf{R}_{\mathbf{Y}} = \mathbf{A}\mathbf{R}_{\mathbf{X}}\mathbf{A}' + (\mathbf{A}\boldsymbol{\mu}_{\mathbf{X}})\mathbf{b}' + \mathbf{b}(\mathbf{A}\boldsymbol{\mu}_{\mathbf{X}})' + \mathbf{b}\mathbf{b}'$$

Autocovariance

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{A}\mathbf{C}_{\mathbf{X}}\mathbf{A}' \quad \text{and} \quad \mathbf{C}_{\mathbf{X}} = \mathbf{A}^{-1}\mathbf{C}_{\mathbf{Y}}(\mathbf{A}^{-1})'$$

Linear Transform of a Random Vector

Cross-correlation

$$\mathbf{R_{XY}} = \mathbf{R_X}\mathbf{A}' + \boldsymbol{\mu}_\mathbf{X}\mathbf{b}'$$

Cross-covariance

$$\mathbf{C_{XY}} = \mathbf{C_X}\mathbf{A}'$$

Example 8.8

Given the expected value $\boldsymbol{\mu}_{\mathbf{X}}$, the correlation $\mathbf{R}_{\mathbf{X}}$, and the covariance $\mathbf{C}_{\mathbf{X}}$ of random vector $\mathbf{X}$ in Example 8.7, and $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}, \tag{8.35}$$

find the expected value $\boldsymbol{\mu}_{\mathbf{Y}}$, the correlation $\mathbf{R}_{\mathbf{Y}}$, and the covariance $\mathbf{C}_{\mathbf{Y}}$.

From the matrix operations of Theorem 8.8, we obtain $\boldsymbol{\mu}_{\mathbf{Y}} = \begin{bmatrix} 1/3 & 2 & 3 \end{bmatrix}'$ and

$$\mathbf{R}_{\mathbf{Y}} = \begin{bmatrix} 1/6 & 13/12 & 4/3 \\ 13/12 & 7.5 & 9.25 \\ 4/3 & 9.25 & 12.5 \end{bmatrix}; \qquad \mathbf{C}_{\mathbf{Y}} = \begin{bmatrix} 1/18 & 5/12 & 1/3 \\ 5/12 & 3.5 & 3.25 \\ 1/3 & 3.25 & 3.5 \end{bmatrix}. \tag{8.36}$$

The calculations are not short and simple...

Example 8.9

Continuing Example 8.8 for random vectors $\mathbf{X}$ and $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, calculate

(a) The cross-correlation matrix $\mathbf{R}_{\mathbf{XY}}$ and the cross-covariance matrix $\mathbf{C}_{\mathbf{XY}}$.

(b) The correlation coefficients ρ_{Y_1,Y_3} and ρ_{X_2,Y_1}.

Gaussian Random Vectors

- Multiple Gaussian random variables appear in many practical applications of probability theory.
- The **multivariate Gaussian distribution** is a probability model for n random variables with the property that the marginal PDFs are all Gaussian.
- A set of random variables described by the multivariate Gaussian PDF is said to be *jointly Gaussian*.
- A vector whose components are jointly Gaussian random variables is said to be a **Gaussian random vector**.

Gaussian Random Vectors

$\mathbf{X}$ is the **Gaussian** $(\boldsymbol{\mu}_{\mathbf{X}}, \mathbf{C}_{\mathbf{X}})$ random vector with expected value $\boldsymbol{\mu}_{\mathbf{X}}$ and covariance $\mathbf{C}_{\mathbf{X}}$ if and only if

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \, det(\mathbf{C}_{\mathbf{X}})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}})' \mathbf{C}_{\mathbf{X}}^{-1} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}})\right)$$

where $det(\mathbf{C}_{\mathbf{X}})$, the determinant of $\mathbf{C}_{\mathbf{X}}$ satisfies $det(\mathbf{C}_{\mathbf{X}}) > \mathbf{0}$, which means that no random variable X_i is a linear combination of the other random variables in $\mathbf{X}$, i.e. each contributes uniquely.

n is the number of random variables in $\mathbf{X}$.

Gaussian Random Vectors

$\mathbf{X}$ is the **Gaussian** $(\boldsymbol{\mu}_{\mathbf{X}}, \mathbf{C}_{\mathbf{X}})$ random vector with expected value $\boldsymbol{\mu}_{\mathbf{X}}$ and covariance $\mathbf{C}_{\mathbf{X}}$ if and only if

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \, det(\mathbf{C}_{\mathbf{X}})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}})' \mathbf{C}_{\mathbf{X}}^{-1} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}})\right)$$

- If $n = 1$ then we simply have the definition of before, the 1-dimensional **Gaussian** (μ, σ^2).
- If $n = 2$, we have the **Bivariate Gaussian PDF**.
  - By setting $\mathbf{X} = [X \; Y]'$ and $\mathbf{x} = [x \; y]'$ the formula can be derived.

Gaussian Random Vectors

A Gaussian random vector $\mathbf{X}$ has *independent* components if and only if $\mathbf{C_X}$ is a diagonal matrix.

(That is $\mathbf{Cov}[X_i, X_j] = 0$ for all $i \neq j$, and $\mathbf{C_X} = \mathbf{diag}[\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2]$)

Remember that if X and Y are independent, then $\mathbf{Cov}[X, Y] = 0$

Example 8.10

Consider the outdoor temperature at a certain weather station. On May 5, the temperature measurements in units of degrees Fahrenheit taken at 6 AM, 12 noon, and 6 PM are all Gaussian random variables, X_1, X_2, X_3, with variance 16 degrees^2. The expected values are 50 degrees, 62 degrees, and 58 degrees respectively. The covariance matrix of the three measurements is

$$\mathbf{C_X} = \begin{bmatrix} 16.0 & 12.8 & 11.2 \\ 12.8 & 16.0 & 12.8 \\ 11.2 & 12.8 & 16.0 \end{bmatrix}. \tag{8.45}$$

(a) Write the joint PDF of X_1, X_2 using the algebraic notation of Definition 5.10.

(b) Write the joint PDF of X_1, X_2 using vector notation.

(c) Write the joint PDF of $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$ using vector notation.

...

(a) First we note that X_1 and X_2 have expected values $\mu_1 = 50$ and $\mu_2 = 62$, variances $\sigma_1^2 = \sigma_2^2 = 16$, and covariance $\text{Cov}[X_1, X_2] = 12.8$. It follows from Definition 5.6 that the correlation coefficient is

$$\rho_{X_1,X_2} = \frac{\text{Cov}\left[X_1, X_2\right]}{\sigma_1\sigma_2} = \frac{12.8}{16} = 0.8. \tag{8.46}$$

From Definition 5.10, the joint PDF is

$$f_{X_1,X_2}(x_1, x_2) = \frac{\exp\left[-\dfrac{(x_1-50)^2 - 1.6(x_1-50)(x_2-62) + (x_2-62)^2}{19.2}\right]}{60.3}.$$

(b) Let $\mathbf{W} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$ denote a vector representation for random variables X_1 and X_2. From the covariance matrix $\mathbf{C_X}$, we observe that the 2×2 submatrix in the upper left corner is the covariance matrix of the random vector $\mathbf{W}$. Thus

$$\boldsymbol{\mu}_\mathbf{W} = \begin{bmatrix} 50 \\ 62 \end{bmatrix}, \qquad \mathbf{C_W} = \begin{bmatrix} 16.0 & 12.8 \\ 12.8 & 16.0 \end{bmatrix}. \tag{8.47}$$

We observe that $\det(\mathbf{C_W}) = 92.16$ and $\det(\mathbf{C_W})^{1/2} = 9.6$. From Definition 8.12, the joint PDF of $\mathbf{W}$ is

$$f_\mathbf{W}(\mathbf{w}) = \frac{1}{60.3} \exp\left(-\frac{1}{2}(\mathbf{w} - \boldsymbol{\mu}_\mathbf{W})^T \mathbf{C}_\mathbf{W}^{-1} (\mathbf{w} - \boldsymbol{\mu}_\mathbf{W})\right). \tag{8.48}$$

(c) Since $\boldsymbol{\mu}_\mathbf{X} = \begin{bmatrix} 50 & 62 & 58 \end{bmatrix}'$ and $\det(\mathbf{C_X})^{1/2} = 22.717$, $\mathbf{X}$ has PDF

$$f_\mathbf{X}(\mathbf{x}) = \frac{1}{357.8} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_\mathbf{X})^T \mathbf{C}_\mathbf{X}^{-1} (\mathbf{x} - \boldsymbol{\mu}_\mathbf{X})\right). \tag{8.49}$$

Gaussian Random Vectors

Given an n-dimensional Gaussian random vector $\mathbf{X}$ with expected value $\boldsymbol{\mu}_{\mathbf{X}}$ and covariance $\mathbf{C}_{\mathbf{X}}$, and an $m \times n$ matrix $\mathbf{A}$ with $\mathbf{rank}(\mathbf{A}) = m$,

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$$

is an m-dimensional Gaussian random vector with expected value $\boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{b}$ and covariance $\mathbf{C}_{\mathbf{Y}} = \mathbf{A}\mathbf{C}_{\mathbf{X}}\mathbf{A}'$.

Gaussian Random Vectors

We don't cover the **Standard Normal Random Vector**

Example 8.11

Continuing Example 8.10, use the formula $Y_i = (5/9)(X_i - 32)$ to convert the three temperature measurements to degrees Celsius.

(a) What is $\boldsymbol{\mu}_{\mathbf{Y}}$, the expected value of random vector $\mathbf{Y}$?

(b) What is $\mathbf{C}_{\mathbf{Y}}$, the covariance of random vector $\mathbf{Y}$?

(c) Write the joint PDF of $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix}'$ using vector notation.

(a) In terms of matrices, we observe that $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 5/9 & 0 & 0 \\ 0 & 5/9 & 0 \\ 0 & 0 & 5/9 \end{bmatrix}, \qquad \mathbf{b} = -\frac{160}{9}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{8.55}$$

(b) Since $\boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} 50 & 62 & 58 \end{bmatrix}'$, from Theorem 8.11,

$$\boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{b} = \begin{bmatrix} 10 \\ 50/3 \\ 130/9 \end{bmatrix}. \tag{8.56}$$

(c) The covariance of $\mathbf{Y}$ is $\mathbf{C}_{\mathbf{Y}} = \mathbf{A}\mathbf{C}_{\mathbf{X}}\mathbf{A}'$. We note that $\mathbf{A} = \mathbf{A}' = (5/9)\mathbf{I}$ where $\mathbf{I}$ is the 3×3 identity matrix. Thus $\mathbf{C}_{\mathbf{Y}} = (5/9)^2\mathbf{C}_{\mathbf{X}}$ and $\mathbf{C}_{\mathbf{Y}}^{-1} = (9/5)^2\mathbf{C}_{\mathbf{X}}^{-1}$. The PDF of $\mathbf{Y}$ is

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{24.47}\exp\left(-\frac{81}{50}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}})^T\mathbf{C}_{\mathbf{X}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{Y}})\right). \tag{8.57}$$