Systems and Signals 344

2025

Lecture 19

Conditional Probability Models: Conditioning by an Event

Sections 7.1-7.3

Conditional Probability Models

In many applications of probability, we have a probability model of an experiment but it is impossible to observe the outcome of the experiment directly. Instead we observe an event that is related to the outcome.

It could also be that we obtain information about a random variable before it is possible to observe the random variable itself.

In these situations, we obtain a **conditional probability model** by modifying the original probability model to take into account the *information gained* from the event we have observed.

THE AUTHOR OF THE WINDOWS FILE COPY DIALOG VISITS SOME FRIENDS.

Recap

Conditional Probability

The conditional probability of the event A given the occurrence of the event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditioning a Random Variable by an Event

The conditional PMF $P_{X|B}(x)$ and conditional PDF $f_{X|B}(x)$ are probability models that use the definition of conditional probability, to incorporate partial knowledge of the outcome of an experiment. The partial knowledge is that the outcome is $X \in B \subset S_X$.

Conditioning a Random Variable by an Event

- Now, instead asking about some event A given some other event B or $P(A|B)$, we now ask a question related to *random variable* X given some event B. Therefore, we now write A in terms of X , for example $P(X \leq x|B)$.
- If X is *discrete*, then typically $A = \{X = x\}$ form some value x.
- If X is *continuous*, then typically $A = \{x_1 < X \leq x_2\}$ or $A = \{x < X \leq x + dx\}$

Conditioning a Random Variable by an Event

Given the event B with $P(B) > 0$, the **conditional cumulative distribution function (CDF)** of X is

$$F_{X|B}(x) = P(X \leq x|B) = \frac{P(X \leq x \cap B)}{P(B)} = \frac{P(B|X \leq x) \cdot F_X(x)}{P(B)}$$

This applies to discrete, continuous, and mixed random variables.

Conditioning a Random Variable by an Event

Given the event B with $P(B) > 0$, the **conditional probability mass function (PMF)** of *discrete* random variable X is

$$P_{X|B}(x) = P(X = x|B)$$

Given the event B with $P(B) > 0$, the **conditional probability density function (PDF)** of *continuous* random variable X is

$$f_{X|B}(x) = \frac{dF_{X|B}(x)}{dx}$$

Conditioning a Random Variable by an Event

This simplifies when considering whether $x \in B$. For a random variable X and an event $B \subset S_X$ with $P(B) > 0$, the conditional PMF/PDF of X given B is:

Discrete:

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P(B)} & x \in B \\ 0 & \text{otherwise} \end{cases}$$

Continuous:

$$f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{P(B)} & x \in B \\ 0 & \text{otherwise} \end{cases}$$

Conditioning a Random Variable by an Event

When $B \subset S_X$, we sometimes just write it directly in terms of X.

For example, if $B = X \geq \mathbf{5}$, then we can write $P_{X|X \geq 5}(x)$

Example 7.2

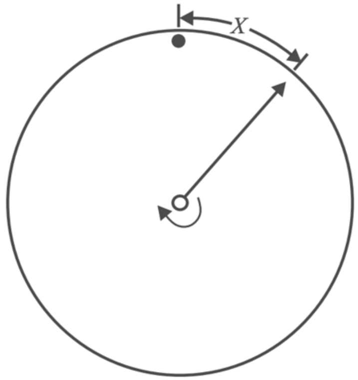
A website distributes instructional videos on bicycle repair. The length of a video in minutes X has PMF

$$P_X(x) = \begin{cases} 0.15 & x = 1, 2, 3, 4, \\ 0.1 & x = 5, 6, 7, 8, \\ 0 & \text{otherwise.} \end{cases} \tag{7.3}$$

Suppose the website has two servers, one for videos shorter than five minutes and the other for videos of five or more minutes. What is the PMF of video length in the second server?

Example 7.3

For the pointer-spinning experiment of Example 4.1, find the conditional PDF of the pointer position for spins in which the pointer stops on the left side of the circle.



Conditioning a Random Variable by an Event

For random variable X resulting from an experiment with *partition* $B_1, \ldots, B_m$

Discrete

$$P_X(x) = \sum_{i=1} P_{X|B_i}(x) P(B_i)$$

Continuous

$$f_X(x) = \sum_{i=1} f_{X|B_i}(x) P(B_i)$$

Example 7.6

Let X denote the number of additional years that a randomly chosen 70-year-old person will live. If the person has high blood pressure, denoted as event H, then X is a geometric $(p = 0.1)$ random variable. Otherwise, if the person's blood pressure is normal, event N, X has a geometric $(p = 0.05)$ PMF. Find the conditional PMFs $P_{X|H}(x)$ and $P_{X|N}(x)$. If 40 percent of all 70-year-olds have high blood pressure, what is the PMF of X?

Conditioning a Random Variable by an Event

The Axioms of Probability hold for Conditional Probability as well

Discrete X	**Continuous X**
For any $x \in B$, $P_{X\|B}(x) \geq 0$	For any $x \in B$, $f_{X\|B}(x) \geq 0$
$\sum_{x \in B} P_{X\|B}(x) = 1$	$\int_B f_{X\|B}(x)\, dx = 1$
The conditional probability that X is in the set C is: $P(C\|B) = \sum\limits_{x \in C} P_{X\|B}(x)$	The conditional probability that X is in the set C is: $P(C\|B) = \int\limits_C f_{X\|B}(x)\, dx$

Conditional Expected Value Given an Event

The definitions of conditional expected value $\mathrm{E}[X|B]$ and conditional variance $\mathbf{Var}[X|B]$ correspond to the definitions of $\mathrm{E}[X]$ and $\mathbf{Var}[X]$ with $P_{X|B}(x)$ replacing $P_X(x)$ or $f_{X|B}(x)$ replacing $f_X(x)$.

Conditional Expected Value Given an Event

The **conditional expected value** of random variable X given condition B is:

Discrete

$$\mathrm{E}[X|B] = \mu_{X|B} = \sum_{x \in B} x P_{X|B}(x)$$

Continuous

$$\mathrm{E}[X|B] = \mu_{X|B} = \int_{-\infty}^{\infty} x f_{X|B}(x)\, dx$$

Conditional Expected Value Given an Event

For a random variable X resulting from an experiment with partition $B_1, \ldots, B_m$,

$$\mathrm{E}[X] = \sum_{i=1}^{m} \mathrm{E}[X|B_i]P(B_i)$$

Conditional Expected Value Given an Event

The **conditional expected value** of $Y = g(X)$ given condition B is:

Discrete

$$\mathrm{E}[Y|B] = \mathrm{E}[g(X)|B] = \sum_{x \in B} g(x) P_{X|B}(x)$$

Continuous

$$\mathrm{E}[Y|B] = \mathrm{E}[g(X)|B] = \int_{-\infty}^{\infty} g(x) f_{X|B}(x)\, dx$$

Conditional Expected Value Given an Event

The **conditional variance** of X given event B is

$$\mathrm{Var}[X|B] = \mathrm{E}\left[(X - \mu_{X|B})^2 | B\right] = \mathrm{E}[X^2|B] - \mu_{X|B}^2$$

The **conditional standard deviation** is $\sigma_{X|B} = \sqrt{\mathrm{Var}[X|B]}$.

Conditional Expected Value Given an Event

The **conditional variance** of X given event B is

$$\mathrm{Var}[X|B] = \mathrm{E}\left[(X - \mu_{X|B})^2|B\right] = \mathrm{E}[X^2|B] - \mu_{X|B}^2$$

The **conditional standard deviation** is $\sigma_{X|B} = \sqrt{\mathrm{Var}[X|B]}$.

If, for example, $\sigma_{X|B}$ is smaller than σ_X then we can say that learning the occurrence of B reduces our uncertainty about X because it shrinks the range of typical values of X

Example 7.8

Find the conditional expected value, the conditional variance, and the conditional standard deviation for the long videos defined in Example 7.2.

$$\mathrm{E}[X|L] = \mu_{X|L} = \sum_{x=5}^{8} x P_{X|L}(x) = 0.25 \sum_{x=5}^{8} x = 6.5 \text{ minutes.}$$

$$\mathrm{E}\left[X^2|L\right] = 0.25 \sum_{x=5}^{8} x^2 = 43.5 \text{ minutes}^2.$$

$$\mathrm{Var}[X|L] = \mathrm{E}\left[X^2|L\right] - \mu_{X|L}^2 = 1.25 \text{ minutes}^2.$$

$$\sigma_{X|L} = \sqrt{\mathrm{Var}[X|L]} = 1.12 \text{ minutes.}$$

Note that $\sigma_X = 2.256$ min

Conditioning Two RVs by an Event

An experiment produces two random variables, X and Y . We learn that the outcome (x, y) is an element of an event, B. We use the information $(x, y) \in B$ to construct a new probability model.

This new model is called the **conditional joint PMF/PDF**,

Conditioning Two RVs by an Event

Given the event B with $P(B) > 0$, the **conditional joint PMF** of *discrete* random variables X and Y is

$$P_{X,Y|B}(x, y) = P(X = x, Y = y|B)$$

For any event B, a region of the X, Y plane with $P(B) > 0$,

$$P_{X,Y|B}(x, y) = \begin{cases} \frac{P_{X,Y}(x,y)}{P(B)} & (x, y) \in B \\ 0 & \text{otherwise} \end{cases}$$

Conditioning Two RVs by an Event

Given the event B with $P(B) > 0$, the **conditional joint PDF** of *continuous* random variables X and Y is

$$f_{X,Y|B}(x, y) = \frac{\partial^2 F_{X,Y|B}(x,y)}{\partial x \partial y}$$

For any event B, a region of the x-y plane with $P(B) > 0$,

$$f_{X,Y|B}(x, y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P(B)} & (x, y) \in B \\ 0 & \text{otherwise} \end{cases}$$

Example 7.10

X and Y are random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \leq x \leq 5, 0 \leq y \leq 3, \\ 0 & \text{otherwise.} \end{cases} \tag{7.22}$$

Find the conditional PDF of X and Y given the event $B = \{X + Y \geq 4\}$.

Conditioning Two RVs by an Event

For random variables X and Y and an event B of nonzero probability, the conditional expected value of $W = g(X, Y)$ given B is

Discrete

$$\mathrm{E}[W|B] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y|B}(x, y)$$

Continuous

$$\mathrm{E}[W|B] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y|B}(x, y)\, dx\, dy$$

Example

Continuing Example 7.10, find the conditional expected value and the conditional variance of $W = XY$ given the event $B = \{X + Y \geq 4\}$