Systems and Signals 344

2025

Lecture 18

Probability Models of Derived Random Variables

Selections from Chapter 6

Intro: Probability Models of Derived RVs

- There are many situations in which we observe one or more random variables and use their values to compute a new random variable.
    - If X is voltage over a resistor (r), then the power dissipated in the resistor is $Y = \frac{X^2}{r}$
    - If X is the amplitude of a transmitted signal, Y is the distance to the receiver. Then amplitude at receiver can be $W = \frac{X}{Y}$.

Intro: Probability Models of Derived RVs

We will only look at situations containing one or two random variables.

1. Perform experiment, observe X, get $P_X(x)$ or $f_X(x)$, then calculate $W = g(X)$
2. Perform experiment, observe X and Y, get $P_{X,Y}(x, y)$ or $f_{X,Y}(x, y)$, then calculate $W = g(X, Y)$

Remember we can calculate things like $\mathbf{E}[W]$ directly from (for example) $f_X(x)$ and $g(X)$, without the need to first calculate $f_W(w)$

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PMF of a Functions of a Single Discrete RV

Steps:

1. Perform experiment and observe an outcome s.
2. From s, find x, the corresponding value of random variable X.
3. Observe y by calculating $y = g(x)$.
4. For all possible value of $x \in S_X$ we compute $y = g(x)$ to get S_Y.

For a discrete random variable X, the PMF of $Y = g(X)$ is

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

(sum all the probabilities of x where $g(x) = y$)

PMF of a Functions of a Single Discrete RV

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(sum all the probabilities of x where $g(x) = y$)

If $g(X)$ transforms different values of x into different values of y (i.e. $g(x_i) \neq g(x_j)$ if $x_i \neq x_j$) then:

$$P_Y(y) = P[Y = g(x)] = P[X = x] = P_X(x)$$

PMF of a Function of Two Discrete RVs

$P_W(w)$, *the PMF of a function of two discrete random variables* X *and* Y *is the sum of the probabilities of all sample values* (x, y) *for which* $g(x, y) = w$.

Note that when X and Y are discrete random variables, S_W, the range of W, is a countable set corresponding to all possible values of $g(X, Y)$.

PMF of a Function of Two Discrete RVs

For discrete random variables X and Y, the derived random variable $W = g(X, Y)$ has PMF

$$P_W(w) = \sum_{(x,y):g(x,y)=w} P_{X,Y}(x, y)$$

Example 6.1

$P_{L,X}(l,x)$	$x = 40$	$x = 60$
$l = 1$	0.15	0.1
$l = 2$	0.3	0.2
$l = 3$	0.15	0.1

A firm sends out two kinds of newsletters. One kind contains only text and grayscale images and requires 40 cents to print each page. The other kind contains color pictures that cost 60 cents per page. Newsletters can be 1, 2, or 3 pages long. Let the random variable L represent the length of a newsletter in pages. $S_L = \{1, 2, 3\}$. Let the random variable X represent the cost in cents to print each page. $S_X = \{40, 60\}$. After observing many newsletters, the firm has derived the probability model shown above. Let $W = g(L, X) = LX$ be the total cost in cents of a newsletter. Find the range S_W and the PMF $P_W(w)$.

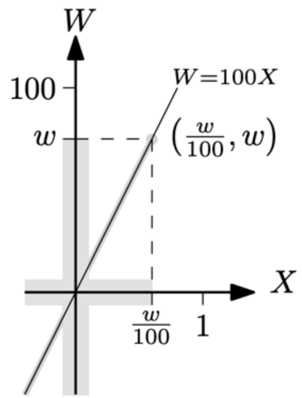
Functions Yielding Continuous RVs

When X and $W = g(X)$ are *continuous* random variables, we develop a *two-step procedure* to derive the PDF $f_W(w)$:

1. Find the CDF $F_W(w) = P(W \leq w) = F_X(g^{-1}(w))$.
2. The PDF is the derivative $f_W(w) = \frac{dF_W(w)}{dw}$.

This procedure *always* works and is easy to remember.

When $g(X)$ is a linear function of X, the method is straightforward. Otherwise, finding $F_W(w)$ can be tricky.



The function $W = 100X$, where X in Example 4.2 is the location of the pointer measured in meters. To find the CDF $F_W(w) = \mathrm{P}[W \leq w]$, the first step is to translate the event $\{W \leq w\}$ into an event described by X. Each outcome of the experiment is mapped to an (X, W) pair on the line $W = 100X$. Thus the event $\{W \leq w\}$, shown with gray highlight on the vertical axis, is the same event as $\{X \leq w/100\}$, which is shown with gray highlight on the horizontal axis. Both of these events correspond in the figure to observing an (X, W) pair along the highlighted section of the line $w = g(X) = 100w$. This translation of the event $W = w$ to an event described in terms of X depends only on the function $g(X)$. Specifically, it does not depend on the probability model for X. From the figure, we see that

$$F_W(w) = \mathrm{P}[W \leq w] = \mathrm{P}[100X \leq w] = \mathrm{P}[X \leq w/100] = F_X(w/100). \tag{6.1}$$

The calculation of $F_X(w/100)$ depends on the probability model for X. For this problem, we recall that Example 4.2 derives the CDF of X,

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \leq x < 1, \\ 1 & x \geq 1. \end{cases} \tag{6.2}$$

From this result, we can use algebra to find

$$F_W(w) = F_X\left(\frac{w}{100}\right) = \begin{cases} 0 & \frac{w}{100} < 0, \\ \frac{w}{100} & 0 \leq \frac{w}{100} < 1, \\ 1 & \frac{w}{100} \geq 1, \end{cases} = \begin{cases} 0 & w < 0, \\ \frac{w}{100} & 0 \leq w < 100, \\ 1 & w \geq 100. \end{cases} \tag{6.3}$$

We take the derivative of the CDF of W over each of the intervals to find the PDF:

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 1/100 & 0 \leq w < 100, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Functions Yielding Continuous RVs

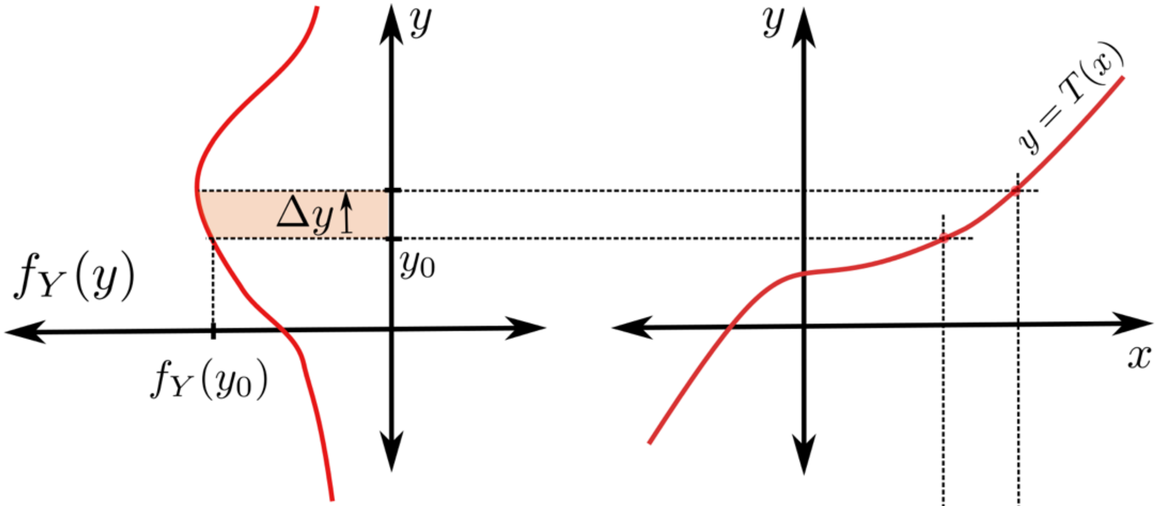
It is also possible to calculate $f_W(w)$ *directly* from $f_X(x)$ when $g(X)$ is **monotonically increasing** or **monotonically decreasing**.

For these we can calculate:

$$f_W(w) = f_X(x)\left(\left|\frac{\partial g(X)}{\partial x}\right|\right)^{-1} = \frac{f_X(x)}{\left|\frac{\partial g(X)}{\partial x}\right|}$$

with $x = g^{-1}(w)$ (substitute x with inverse function)

Remember at the end of the day $f_W(w)$ must be a function of w.

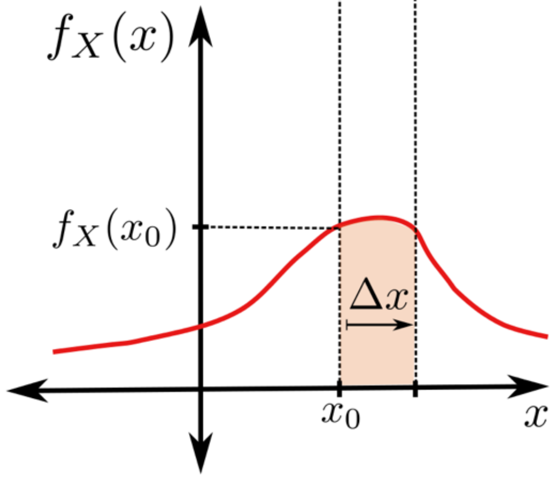


For $\Delta x \to 0^+$

$$f_X(x_0)\,\Delta x \;=\; f_Y(y_0)\,\Delta y$$

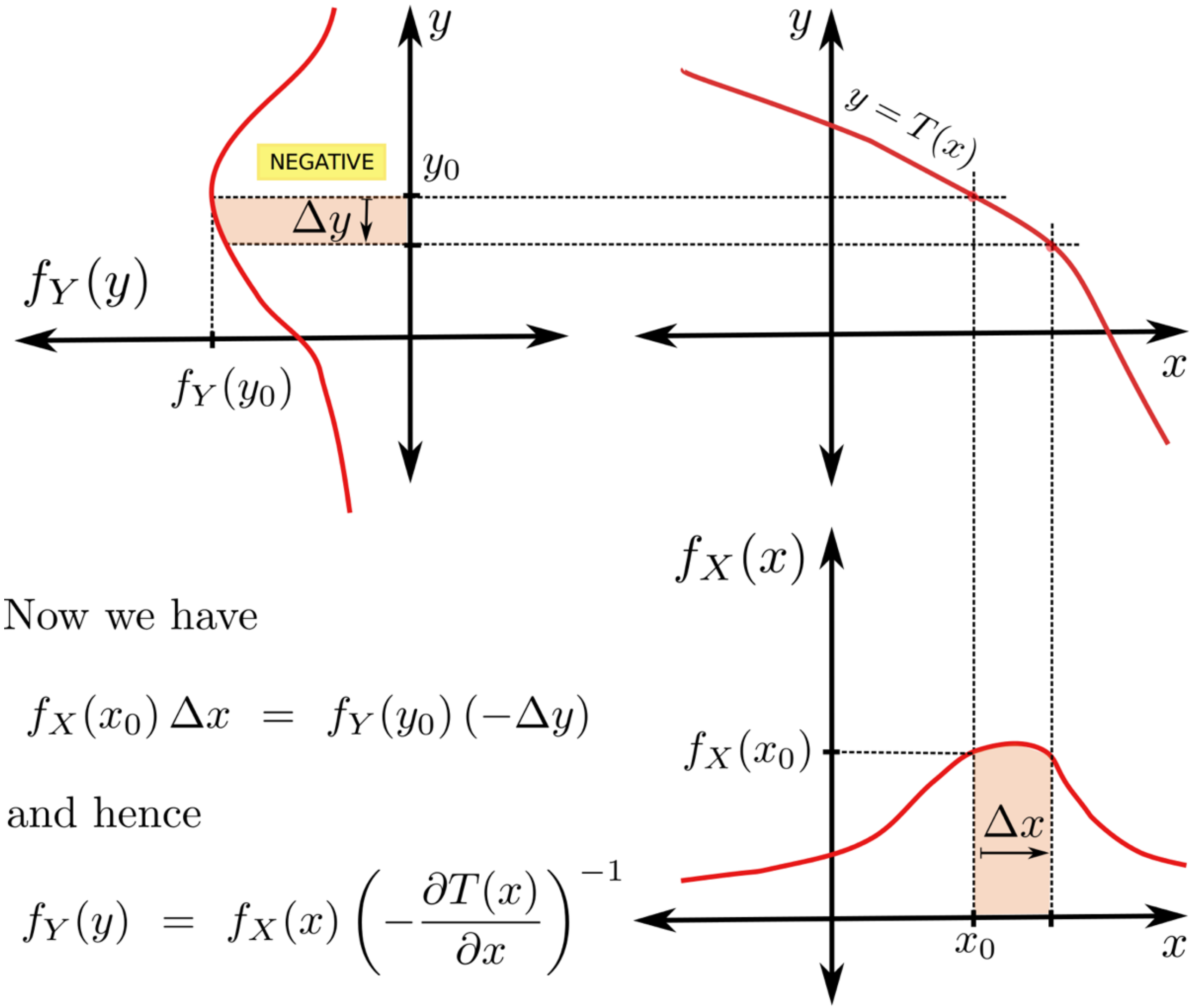
and hence

$$\begin{aligned} f_Y(y_0) &= f_X(x_0)\frac{\Delta x}{\Delta y} \\ &= f_X(x_0)\left(\frac{\Delta y}{\Delta x}\right)^{-1} \\ &= f_X(x_0)\left(\left.\frac{\partial T(x)}{\partial x}\right|_{x=x_0}\right)^{-1} \end{aligned}$$



This is true for any x_0 and so we have $f_Y(y) \;=\; f_X(x)\left(\frac{\partial T(x)}{\partial x}\right)^{-1}$ with $x = T^{-1}(y)$

Remember areas must match



Now we have

$$f_X(x_0)\,\Delta x \;=\; f_Y(y_0)\,(-\Delta y)$$

and hence

$$f_Y(y) \;=\; f_X(x)\left(-\frac{\partial T(x)}{\partial x}\right)^{-1}$$

Since slope is negative, answer stays positive

Functions Yielding Continuous RVs

If $W = aX$, where $a > 0$, then W has CDF and PDF

- $F_W(w) = F_X(\frac{w}{a})$
- $f_W(w) = \frac{1}{a} f_X(\frac{w}{a})$.

This states that multiplying a random variable by a positive constant stretches ($a > 1$) or shrinks ($a < 1$) the original PDF.

If $W = X + b$, then W has CDF and PDF

- $F_W(w) = F_X(w - b)$
- $f_W(w) = f_X(w - b)$.

This states adding a constant to a random variable simply shifts the PDF and CDF.

Example 6.3

The triangular PDF of X is

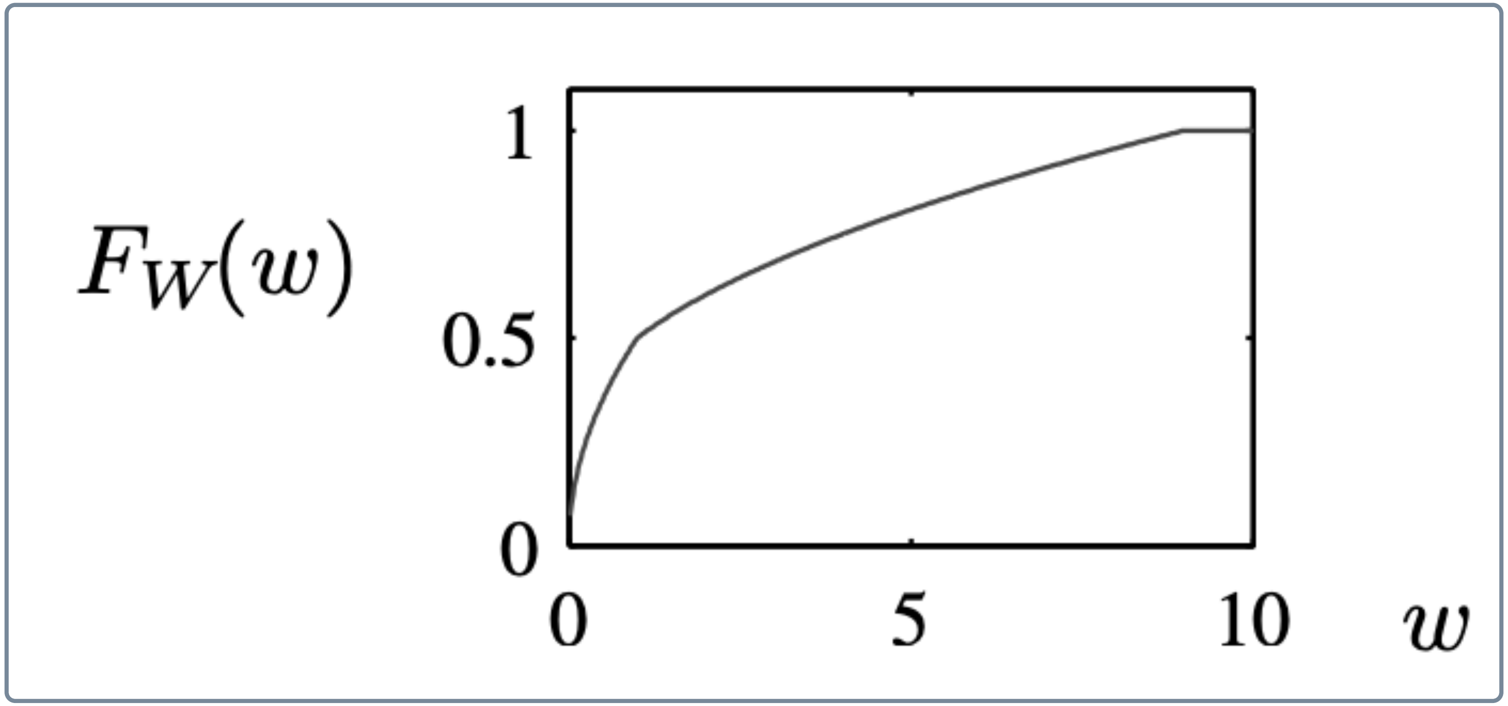
$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

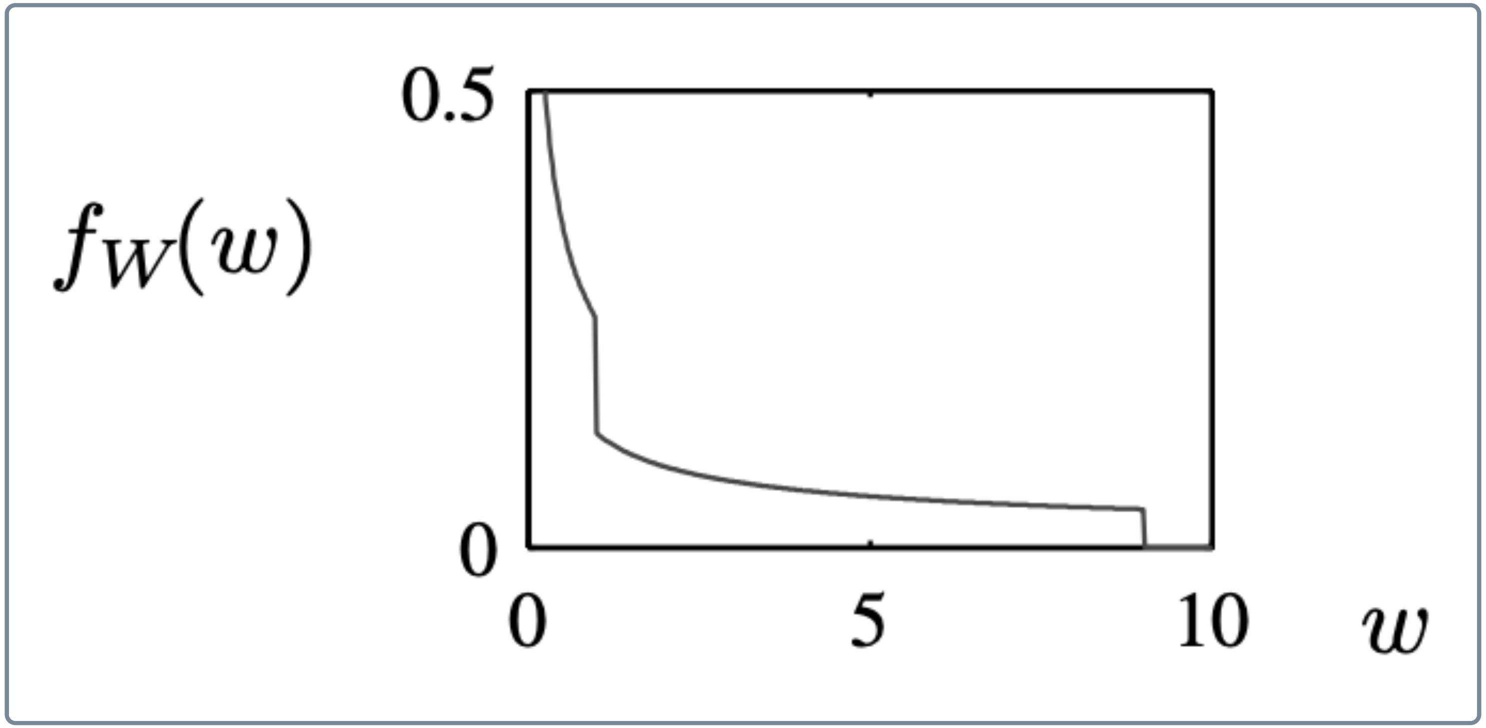
Find the PDF of $W = aX$. Sketch the PDF of W for $a = 1/2, 1, 2$.

Example

Suppose X is the continuous uniform $(-1, 3)$ random variable and $W = X^2$.

Find the CDF $F_W(w)$ and the PDF $f_W(w)$.


$F_W(w)$
1
0.5
0
0
5
10
w


$f_W(w)$
0.5
0
0
5
10
w

Continuous Functions of 2 Continuous RVs

When X and Y are continuous random variables and $g(x, y)$ is a continuous function, $W = g(X, Y)$ is a continuous random variable.

To find the PDF, $f_W(w)$, it is usually helpful to first find the CDF $F_W(w)$ and then calculate the derivative.

To find the CDF:

$$F_W(w) = P(W \leq w) = \int \int_{g(x,y) \leq w} f_{X,Y}(x, y)\, dx\, dy$$

Then to find the PDF:

$$f_W(w) = \frac{dF_W(w)}{dw}$$

Continuous Functions of 2 Continuous RVs

For interest (see example in textbook)

For continuous random variables X and Y , the CDF of $W = \max(X, Y)$ is

$$F_W(w) = F_{X,Y}(w, w) = \int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x, y)\, dx\, dy$$

Example 6.10

X and Y have the joint PDF

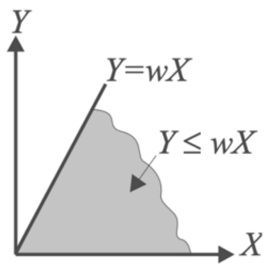
$$f_{X,Y}(x,y) = \begin{cases} \lambda\mu e^{-(\lambda x+\mu y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.34}$$

Find the PDF of $W = Y/X$.

First we find the CDF:

$$F_W(w) = \mathrm{P}[Y/X \leq w] = \mathrm{P}[Y \leq wX]. \tag{6.35}$$

For $w < 0$, $F_W(w) = 0$. For $w \geq 0$, we integrate the joint PDF $f_{X,Y}(x,y)$ over the region of the X, Y plane for which $Y \leq wX$, $X \geq 0$, and $Y \geq 0$ as shown:



$$\begin{aligned} \mathrm{P}[Y \leq wX] &= \int_0^\infty \left(\int_0^{wx} f_{X,Y}(x,y)\, dy \right) dx \\ &= \int_0^\infty \lambda e^{-\lambda x} \left(\int_0^{wx} \mu e^{-\mu y}\, dy \right) dx \\ &= \int_0^\infty \lambda e^{-\lambda x} \left(1 - e^{-\mu w x}\right) dx \\ &= 1 - \frac{\lambda}{\lambda + \mu w}. \end{aligned} \tag{6.36}$$

Therefore,

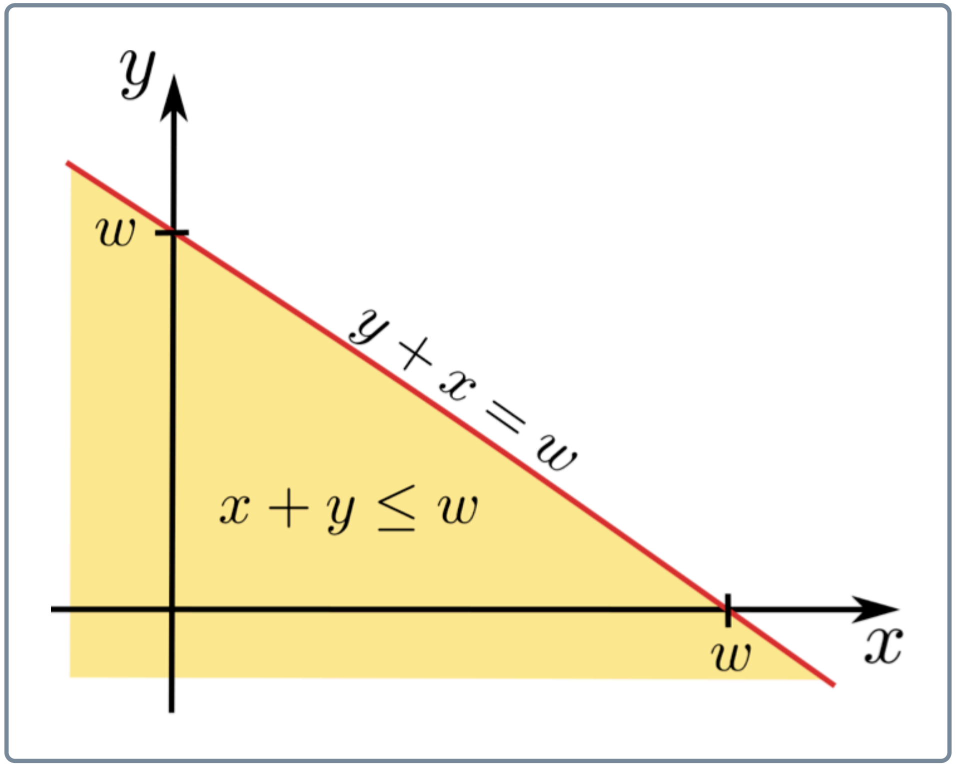
$$F_W(w) = \begin{cases} 0 & w < 0, \\ 1 - \dfrac{\lambda}{\lambda + \mu w} & \omega \geq 0. \end{cases} \tag{6.37}$$

Differentiating with respect to w, we obtain

$$f_W(w) = \begin{cases} \dfrac{\lambda\mu}{(\lambda + \mu w)^2} & w \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.38}$$

PDF of the Sum of Two RVs

Now we examine the sum $W = X + Y$ of two continuous RVs.



$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w - x)dx = \int_{-\infty}^{\infty} f_{X,Y}(w - y, y)dy$$

PDF of the Sum of Two RVs

Recall that for two independent random variables X, and Y, the joint PDF is: $f_{X,Y}(x,y) = f_X(x)f_Y(y)$

When X and Y are *independent* random variables and $W = X + Y$.

For continuous X and Y, the PDF is:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x)f_Y(w-x)dx = \int_{-\infty}^{\infty} f_X(w-y)f_Y(y)dy$$

For discrete X and Y, the PMF is:

$$P_W(w) = \sum_{k=-\infty}^{\infty} P_X(k)P_Y(w-k)$$

PDF of the Sum of Two RVs

When X and Y are *independent* random variables and $W = X + Y$.

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-y)dx = \int_{-\infty}^{\infty} f_X(w-y) f_Y(y)dy$$

$$P_W(w) = \sum_{k=-\infty}^{\infty} P_X(k) P_Y(w-k)$$

This is simply the *convolution* of the two PDF. Therefore:

$$f_W(w) = f_{X+Y}(w) = f_X(w) * f_Y(w)$$

Remember this only applies if X and Y are independent.

Example 6.11

Find the PDF of $W = X + Y$ when X and Y have the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 \leq y \leq 1, 0 \leq x \leq 1, x + y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.43}$$