Systems and Signals 344

2026

Lecture 17

Bivariate Gaussian Random Variables & Multivariate Probability Models

Sections 5.9-5.10

Recap

Gaussian Distribution

X is a **Gaussian** (μ, σ) random variable if the PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Where μ can be any real number and $\sigma > 0$

Note that $\mu = \mu_X$ and $\sigma = \sigma_X$.

Bivariate Gaussian Random Variables

The bivariate Gaussian PDF of X and Y ($f_{X,Y}(x, y)$) has five parameters:

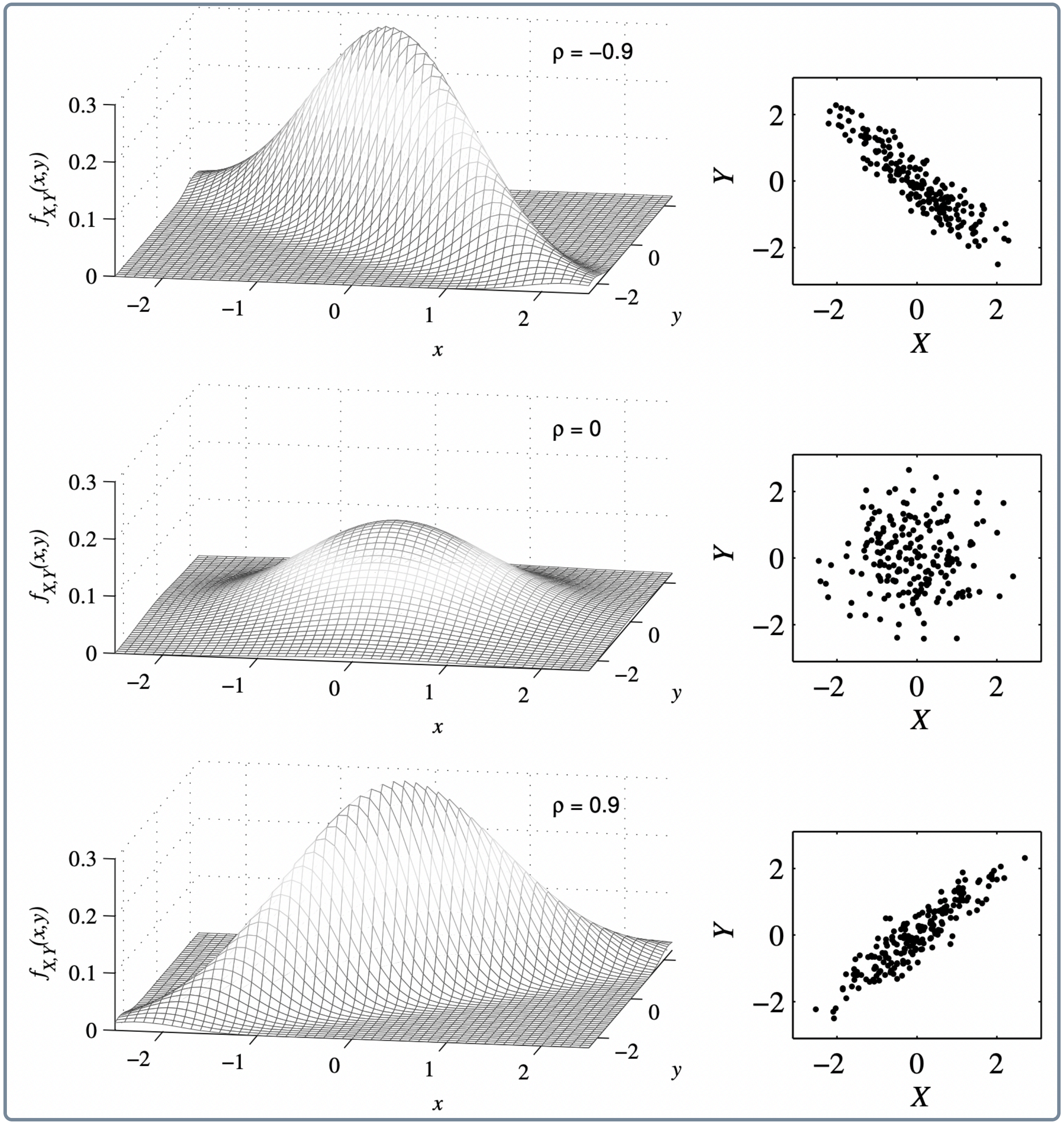
the expected values and standard deviations of X and Y and the correlation coefficient of X and Y ($\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho_{X,Y}$)

The marginal PDF of X and the marginal PDF of Y are both Gaussian.

Bivariate Gaussian Random Variables

Random variables X and Y have a **bivariate Gaussian PDF** with parameters $\mu_X, \mu_Y, \sigma_X > 0, \sigma_Y > 0,$ and $\rho_{X,Y}$, satisfying $-1 < \rho_{X,Y} < 1$ if

$$f_{X,Y}(x,y) = \frac{\exp\left[-\frac{1}{2\left(1-\rho_{X,Y}^2\right)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho_{X,Y}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} - \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right]}{2\pi\,\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$


ρ = −0.9
ρ = 0
ρ = 0.9
$f_{X,Y}(x,y)$
x
y
X
Y

Bivariate Gaussian Random Variables

To examine mathematically the properties of the bivariate Gaussian PDF, we define

$$\tilde{\mu}_Y(x) = \mu_Y + \rho_{X,Y}\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$$

$$\tilde{\sigma}_Y = \sigma_Y\sqrt{1 - \rho^2_{X,Y}}$$

This makes the joint Gaussian PDF:

$$f_{X,Y}(x,y) = \frac{1}{\sigma_X\sqrt{2\pi}}e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \cdot \frac{1}{\tilde{\sigma}_Y\sqrt{2\pi}}e^{-\frac{(y-\tilde{\mu}_Y(x))^2}{2\tilde{\sigma}_Y^2}}$$

Bivariate Gaussian Random Variables

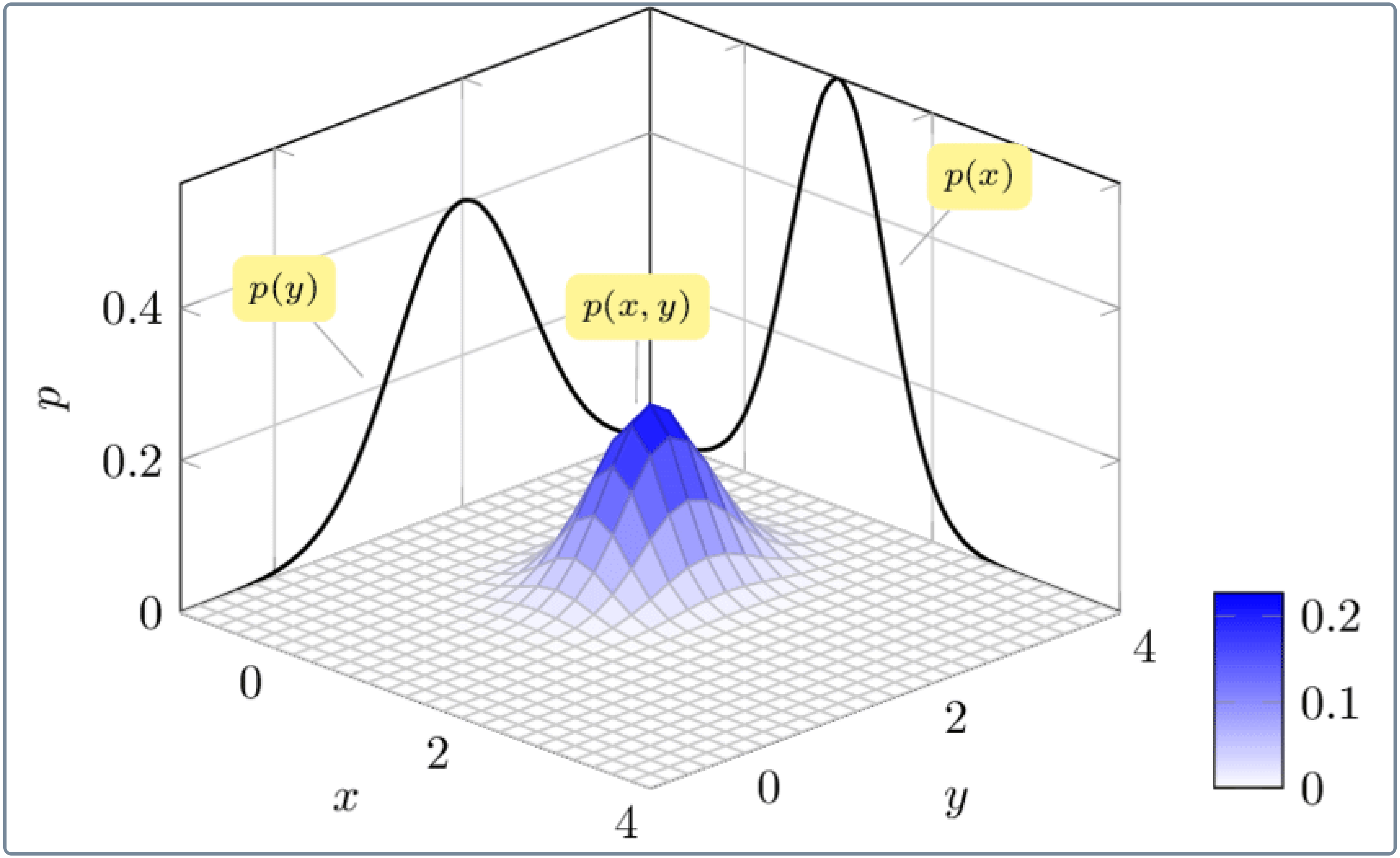
$$f_{X,Y}(x,y) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \cdot \frac{1}{\tilde{\sigma}_Y\sqrt{2\pi}} e^{-\frac{\left(y-\tilde{\mu}_Y(x)\right)^2}{2\tilde{\sigma}_Y^2}}$$

This expresses $f_{X,Y}(x,y)$ as the product of two Gaussian PDFs.

From this we can derive the marginals:

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}$$

$$f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$



Demo

Python Example

Bivariate Gaussian Random Variables

Bivariate Gaussian random variables X and Y have correlation coefficient $\rho_{X,Y}$.

If $\rho_{X,Y} = 0$ (meaning that X and Y are uncorrelated)

Then $f_{X,Y}(x, y) = f_X(x) f_Y(y)$ which means that they are then also independent.

Or stated differently, bivariate Gaussian random variables X and Y are *uncorrelated* if and only if they are *independent*.

Bivariate Gaussian Random Variables

If X and Y are *bivariate Gaussian random variables* and W_1 and W_2 are given by the linearly independent equations

$$W_1 = a_1 X + b_1 Y, \qquad W_2 = a_2 X + b_2 Y,$$

then W_1 and W_2 are bivariate Gaussian random variables such that (for $i = 1, 2$)

$$\begin{aligned} \mathrm{E}[W_i] &= a_i \mu_X + b_i \mu_Y \\ \mathrm{Var}[W_i] &= a_i^2 \sigma_X^2 + b_i^2 \sigma_Y^2 + 2 a_i b_i \rho_{X,Y} \sigma_X \sigma_Y \\ \mathrm{Cov}[W_1, W_2] &= a_1 a_2 \sigma_X^2 + b_1 b_2 \sigma_Y^2 + (a_1 b_2 + a_2 b_1) \rho_{X,Y} \sigma_X \sigma_Y \end{aligned}$$

Example 5.19

For the noisy observation in Example 5.14, find the PDF of $Y = X + Z$.

Example 5.20

Continuing Example 5.19, find the joint PDF of X and Y when $\sigma_X = 4$ and $\sigma_Z = 3$.

Multivariate Probability Models

The probability model of an experiment that produces n random variables can be represented as an n-dimensional CDF.

If all of the random variables are discrete, there is a corresponding n-dimensional PMF.

If all of the random variables are continuous, there is an n-dimensional PDF.

The PDF is the nth partial derivative of the CDF with respect to all n variables.

The probability model (CDF, PMF, or PDF) of n independent random variables is the product of the univariate probability models of the n random variables.

Multivariate Probability Models

Definition 5.11 Multivariate Joint CDF

*The **joint CDF** of* $X_1, \ldots, X_n$ *is*

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \mathrm{P}\left[X_1 \leq x_1, \ldots, X_n \leq x_n\right].$$

Definition 5.12 Multivariate Joint PMF

*The **joint PMF** of the discrete random variables* $X_1, \ldots, X_n$ *is*

$$P_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \mathrm{P}\left[X_1 = x_1, \ldots, X_n = x_n\right].$$

Definition 5.13 Multivariate Joint PDF

*The **joint PDF** of the continuous random variables* $X_1, \ldots, X_n$ *is the function*

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \frac{\partial^n F_{X_1,\ldots,X_n}(x_1, \ldots, x_n)}{\partial x_1 \cdots \partial x_n}.$$

Multivariate Probability Models

Theorem 5.22

If $X_1, \ldots, X_n$ *are discrete random variables with joint PMF* $P_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$,

(a) $P_{X_1,\ldots,X_n}(x_1, \ldots, x_n) \geq 0$,

(b) $\displaystyle\sum_{x_1 \in S_{X_1}} \cdots \sum_{x_n \in S_{X_n}} P_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = 1.$

Theorem 5.23

If $X_1, \ldots, X_n$ *have joint PDF* $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$,

(a) $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) \geq 0$,

(b) $\displaystyle F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(u_1, \ldots, u_n)\, du_1 \cdots du_n,$

(c) $\displaystyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = 1.$

Multivariate Probability Models

For some event A, described in terms of a property of $X_1, \ldots, X_n$.

For example $A := \mathbf{max}_i\, X_i \leq 100$ or
$A := |X_1 + X_2 + \ldots + X_n| \leq 1$

Theorem 5.24

The probability of an event A expressed in terms of the random variables $X_1, \ldots, X_n$ is

$$\textit{Discrete:} \quad \mathrm{P}[A] = \sum_{(x_1,\ldots,x_n) \in A} P_{X_1,\ldots,X_n}(x_1, \ldots, x_n);$$

$$\textit{Continuous:}\ \mathrm{P}[A] = \underset{A}{\int \cdots \int} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n.$$

To calculate **marginals**

Theorem 5.25

For a joint PMF $P_{W,X,Y,Z}(w,x,y,z)$ *of discrete random variables* W,X,Y,Z, *some marginal PMFs are*

$$P_{X,Y,Z}(x,y,z) = \sum_{w \in S_W} P_{W,X,Y,Z}(w,x,y,z),$$

$$P_{W,Z}(w,z) = \sum_{x \in S_X} \sum_{y \in S_Y} P_{W,X,Y,Z}(w,x,y,z),$$

Theorem 5.26

For a joint PDF $f_{W,X,Y,Z}(w,x,y,z)$ *of continuous random variables* W,X,Y,Z, *some marginal PDFs are*

$$f_{W,X,Y}(w,x,y) = \int_{-\infty}^{\infty} f_{W,X,Y,Z}(w,x,y,z)\,dz,$$

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{W,X,Y,Z}(w,x,y,z)\,dw\,dy\,dz.$$

Example 5.22

As in Quiz 5.10, the random variables $Y_1, \ldots, Y_4$ have the joint PDF

$$f_{Y_1,\ldots,Y_4}(y_1,\ldots,y_4) = \begin{cases} 4 & 0 \le y_1 \le y_2 \le 1, 0 \le y_3 \le y_4 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.75}$$

Find the marginal PDFs $f_{Y_1,Y_4}(y_1,y_4)$, $f_{Y_2,Y_3}(y_2,y_3)$, and $f_{Y_3}(y_3)$.

Only calculate $f_{Y_1,Y_4}(y_1,y_4)$ and $f_{Y_1}(y_1)$

Example 5.23

The random variables $X_1, \ldots, X_n$ have the joint PDF

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \begin{cases} 1 & 0 \le x_i \le 1, i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{5.83}$$

Let A denote the event that $\max_i X_i \le 1/2$. Find $\mathrm{P}[A]$.

Multivariate Probability Models

Definition 5.14 N Independent Random Variables

Random variables $X_1, \ldots, X_n$ *are* ***independent*** *if for all* $x_1, \ldots, x_n$,

$$\textit{Discrete:} \quad P_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = P_{X_1}(x_1)\, P_{X_2}(x_2) \cdots P_{X_N}(x_n)\,;$$

$$\textit{Continuous:} \quad f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = f_{X_1}(x_1)\, f_{X_2}(x_2) \cdots f_{X_n}(x_n)\,.$$

Multivariate Probability Models

Suppose we have an experiment consisting of n *independent* subexperiments, in which subexperiment i produces random variable X_i.

If all subexperiments follow the same procedure, then all X_i will have the same PMF or PDF.

We say the random variables X_i are **identically distributed**.

Definition 5.15 **Independent and Identically Distributed (iid)**

$X_1, \ldots, X_n$ *are* ***independent and identically distributed (iid)*** *if*

Discrete: $P_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = P_X(x_1)\,P_X(x_2)\cdots P_X(x_n)\,;$

Continuous: $f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = f_X(x_1)\,f_X(x_2)\cdots f_X(x_n)\,.$