# Systems and Signals 344

# 2026

## Lecture 15

## Joint Probability Density Function & Marginal PDF

Sections 5.4-5.5

# Joint Probability Density Function

*The most useful probability model of continuous random variables $X$ and $Y$ is the joint PDF $f_{X,Y}(x, y)$. It is a generalisation of the PDF of a single random variable.*

# Joint Probability Density Function

The **joint probability density function (PDF)** of the continuous random variables $X$ and $Y$ is the second-order partial derivative of the joint CDF.

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

$$F_{X,Y}(x_1,y_1) = P(X \leq x_1, Y \leq y_1) = \int_{-\infty}^{x_1} \int_{-\infty}^{y_1} f_{X,Y}(x,y)\, dy\, dx$$

For a single random variable $X$, the PDF $f_X(x)$ measures the probability per unit *length*. For two random variables $X, Y$, the joint PDF $f_{X,Y}(x,y)$ measures probability per unit *area*.

# Properties of the Joint PDF

1. **Non-negativity**
   $f_{X,Y}(x, y) \geq 0 \;\; \forall \; x, y$
2. **Unity Volume**
   $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy \, dx = 1$

# Joint Probability between ranges

$$P(x_1 < X \le x_2,\ y_1 < Y \le y_2)$$
$$= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$
$$= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{X,Y}(x, y)\, dx\, dy$$

# Given:

**Example 5.2**

$X$ years is the age of children entering first grade in a school. $Y$ years is the age of children entering second grade. The joint CDF of $X$ and $Y$ is

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 5, \\ 0 & y < 6, \\ (x-5)(y-6) & 5 \le x < 6, 6 \le y < 7, \\ y-6 & x \ge 6, 6 \le y < 7, \\ x-5 & 5 \le x < 6, y \ge 7, \\ 1 & \text{otherwise.} \end{cases} \tag{5.3}$$

Find $F_X(x)$ and $F_Y(y)$.

Using Theorem 5.1(b) and Theorem 5.1(c), we find

$$F_X(x) = \begin{cases} 0 & x < 5, \\ x-5 & 5 \le x < 6, \\ 1 & x \ge 6, \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y < 6, \\ y-6 & 6 \le y < 7, \\ 1 & y \ge 7. \end{cases} \tag{5.4}$$

Referring to Theorem 4.6, we see from Equation (5.4) that $X$ is a continuous uniform $(5,6)$ random variable and $Y$ is a continuous uniform $(6,7)$ random variable.

Derive the joint PDF $f_{X,Y}(x,y)$

# Joint Probability Density Function

The probability that the continuous random variables $(X, Y)$ are in $A$ is:

$$P(A) = \iint\limits_{A} f_{X,Y}(x, y)\, dx\, dy$$

# Marginal PDF

If $X$ and $Y$ are random variables with joint PDF $f_{X,Y}(x,y)$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$F_X(x_1) = F_{X,Y}(x_1,\infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} f_{X,Y}(x,y)\,dx\,dy$$

$$F_Y(y_1) = F_{X,Y}(\infty,y_1) = \int_{-\infty}^{y_1}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

**Example 5.7**

Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \leq x \leq 5, 0 \leq y \leq 3, \\ 0 & \text{otherwise.} \end{cases} \tag{5.19}$$

Find the constant $c$ and $\mathrm{P}[A] = \mathrm{P}[2 \leq X < 3, 1 \leq Y < 3]$.

**Example 5.9**

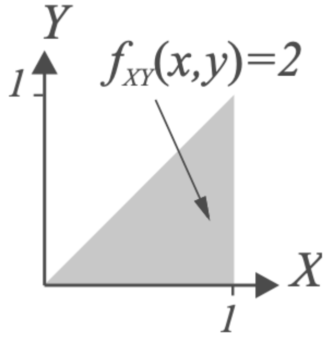
As in Example 5.7, random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 5, 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases} \tag{5.27}$$

What is $\mathrm{P}[A] = \mathrm{P}[Y > X]$?

### Example 5.8

Find the joint CDF $F_{X,Y}(x,y)$ when $X$ and $Y$ have joint PDF




$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.22}$$

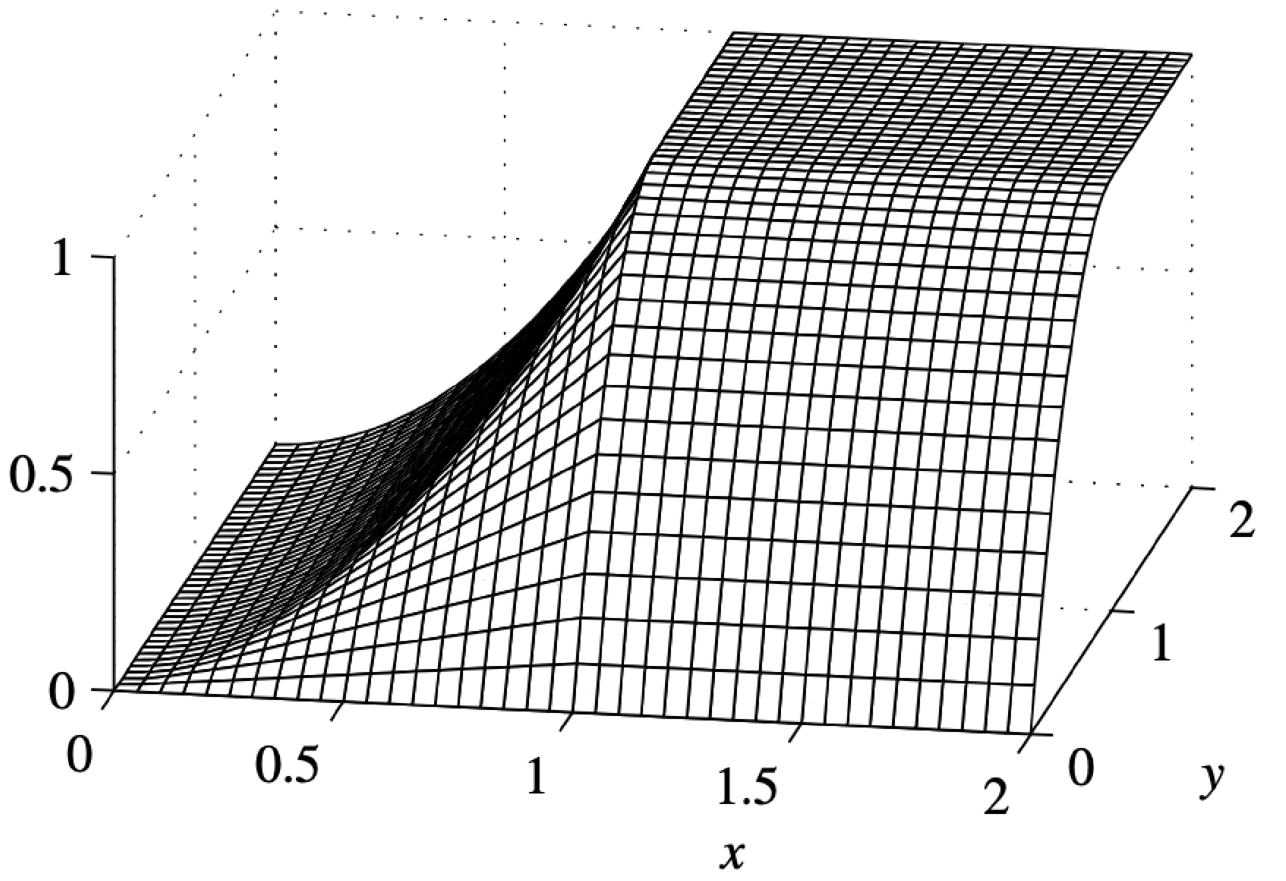



*Figure 5.4* A graph of the joint CDF $F_{X,Y}(x, y)$ of Example 5.8.

**Example 5.10**

The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} 5y/4 & -1 \leq x \leq 1, x^2 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.32}$$

Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.