Systems and Signals 344

2026

Lecture 14

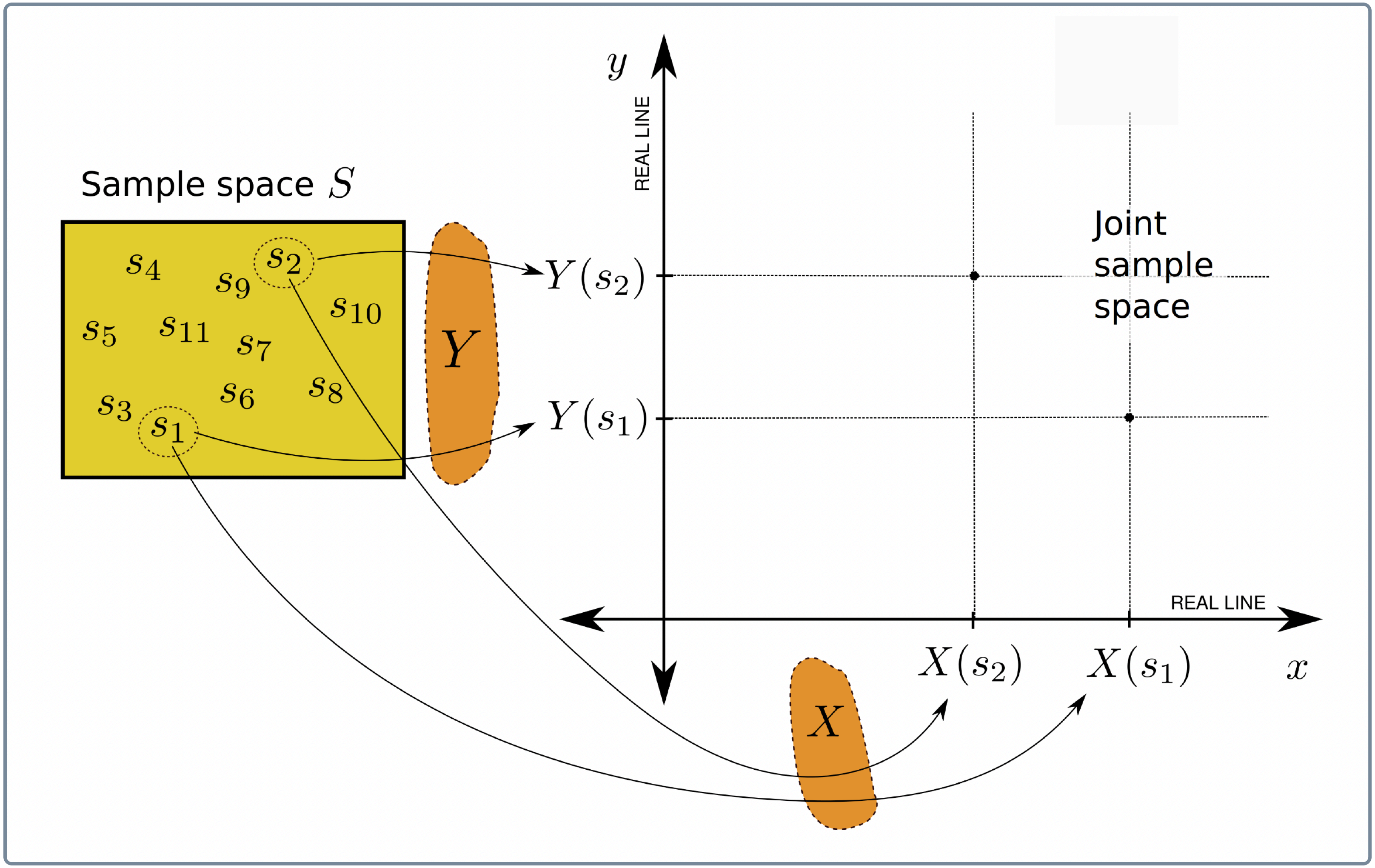
Joint Cumulative Distribution Function & Joint Probability Mass Function & Marginal PMF

Sections 5.1-5.3

Multiple Random Variables

- Up until now we have worked with experiments in which an outcome is mapped to one number.
- From now on, we will analyse experiments in which an outcome is mapped to a collection of numbers (in different Random Variables).
    - Each number is a sample value of a single random variable.
    - We can have experiments producing a collection of random variables, $X_1, X_2, \ldots, X_n$ for any integer n.
    - We will, however, for now focus on $n = 2$ random variables, and call them X and Y.
    - Most formulas do, however, generalise to n random variables.

Multiple Random Variables



An outcome now maps to an (x, y) point on a plane.

Multiple Random Variables

- We also note that a pair of random variables X and Y is the same as the two-dimensional vector $[X\ Y]'$.
- Similar for $X_1, X_2, \ldots, X_n$ random variables, we have a n dimensional vector $\mathbf{X} = [X_1\ X_2\ \ldots\ X_n]'$
- This $\mathbf{X}$ is called a **random vector**.
    - We will cover this topic in depth in Chapter 8.

Multiple Random Variables

- One example situation where this arises is where a signal, X, is emitted by a radio transmitter, and the corresponding signal, Y, eventually arrives at a receiver.
    - We observe Y, but we really want to know X.
    - However, due to noise, Z, we have that $Y = X + Z$, and we must use a probability model to estimate X.
    - Note that due to this relationship we really only need any pair of random variables, (X, Z), (X, Y) or (Y, Z) to analyse experiments related to this system.

Joint Cumulative Distribution Function

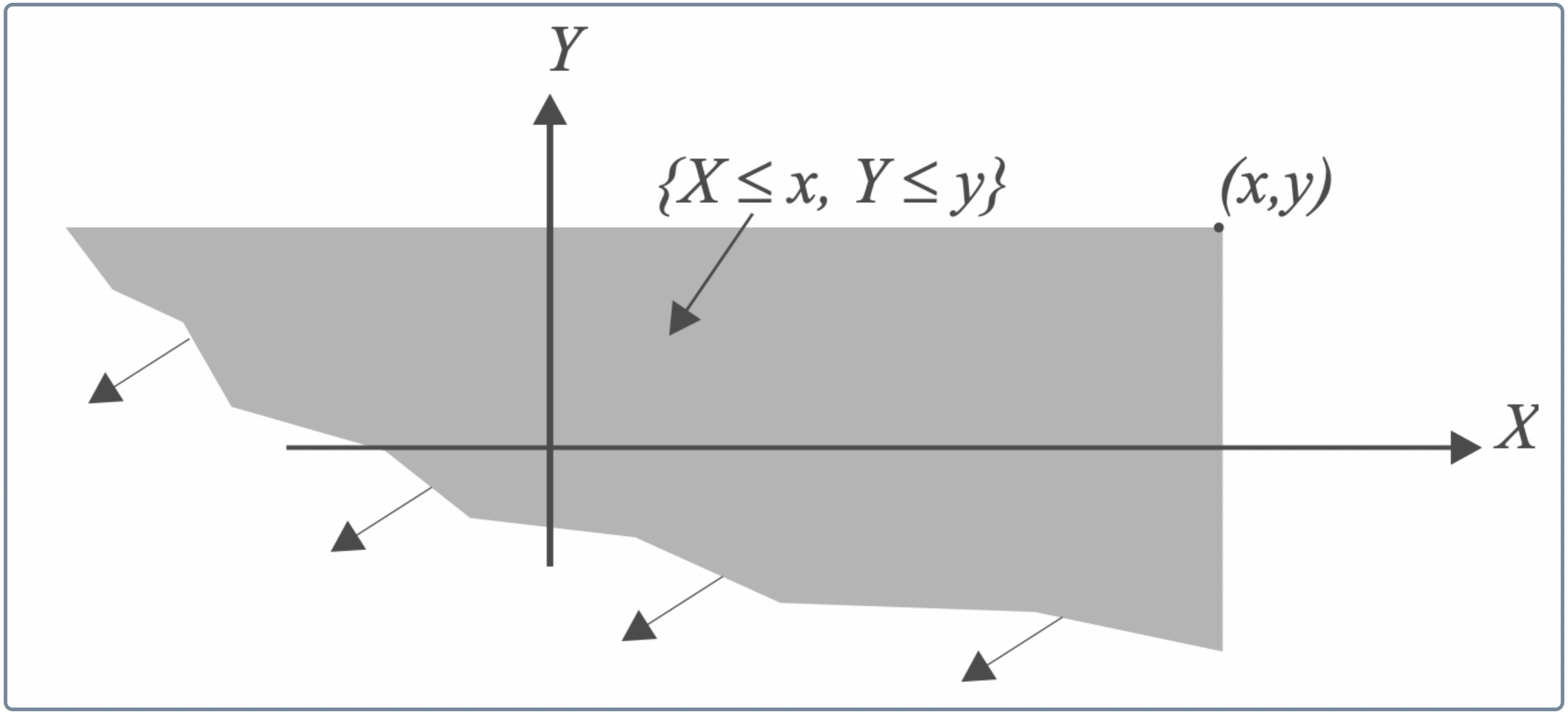
The **Joint cumulative distribution function (CDF)** of random variables X and Y is

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$

$P(X \leq x, Y \leq y)$ is the probability that the value of the random variable X is less than or equal to a particular number x *and* the random variable Y is less than or equal to the particular number y.

Joint Cumulative Distribution Function

- Just as the CDF of one random variable, $F_X(x)$, is the probability of the interval to the left of x, the joint CDF $F_{X,Y}(x, y)$ of two random variables is the probability of the area below and to the left of (x, y).



Joint Cumulative Distribution Function

- Properties of $F_{X,Y}(x,y)$:
  - $F_{X,Y}(-\infty,-\infty)=0$ and $F_{X,Y}(-\infty,y)=0$ and $F_{X,Y}(x,-\infty)=0$ and $F_{X,Y}(\infty,\infty)=1$
  - $0 \le F_{X,Y}(x,y) \le 1$
  - If $x \le x_1$ and $y \le y_1$, then $F_{X,Y}(x,y) \le F_{X,Y}(x_1,y_1)$
    - ($F_{X,Y}(x,y)$ is nondecreasing in x and y)
  - $\lim_{\epsilon \to 0^+} F_{X,Y}(x+\epsilon, y+\epsilon) = F_{X,Y}(x,y)$
    - (continuous from the right for discrete jumps)

Joint Cumulative Distribution Function

- **Marginalisation:**
  - $F_{X,Y}(x, \infty) = F_X(x)$
  - $F_{X,Y}(\infty, y) = F_Y(y)$
  - Here $F_X(x)$ and $F_Y(y)$ are called the **marginal distributions**

Joint Cumulative Distribution Function

The probability that an outcome is in a rectangle in the X, Y plane in terms of the joint CDF is given by

$$(x_1 < X \leq x_2, y_1 < Y \leq y_2) =$$
$$F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$

Example 5.2

X years is the age of children entering first grade in a school. Y years is the age of children entering second grade. The joint CDF of X and Y is

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 5, \\ 0 & y < 6, \\ (x-5)(y-6) & 5 \le x < 6, 6 \le y < 7, \\ y-6 & x \ge 6, 6 \le y < 7, \\ x-5 & 5 \le x < 6, y \ge 7, \\ 1 & \text{otherwise.} \end{cases} \tag{5.3}$$

Find $F_X(x)$ and $F_Y(y)$.

Joint Probability Mass and Density Functions

When the random variables, X and Y are *discrete* we will use a **probability mass function** and if they are *continuous* we will use a **probability density function** as our probability model.

Joint Probability Mass Function

For discrete random variables X and Y, the joint PMF $P_{X,Y}(x, y)$ is the probability that $X = x$ and $Y = y$. It is a complete probability model for X and Y

Joint Probability Mass Function

The **joint probability mass function** of discrete random variables X and Y is

$$P_{X,Y}(x, y) = P(X = x, Y = y)$$

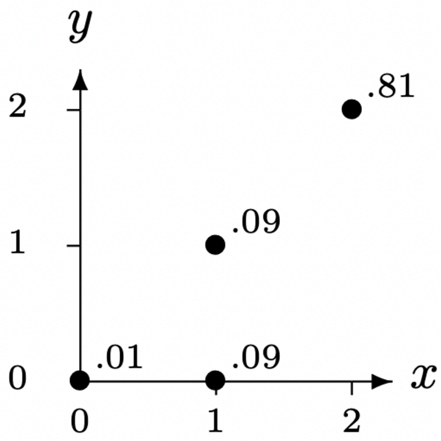
This is the probability of observing the combination of x and y. Remember that $\{X = x, Y = y\}$ is an event in an experiment.

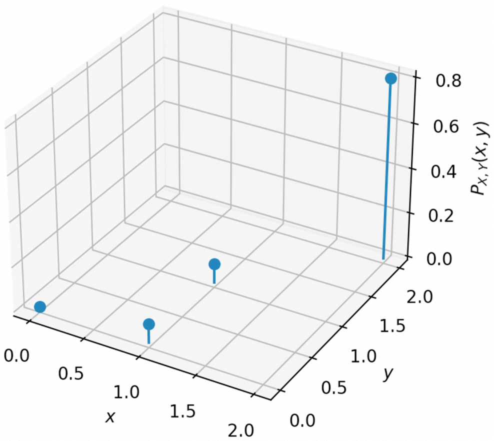
Joint Probability Mass Function

There are three common ways in which a joint PMF is represented: a graph, a list, and a table.

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	0.01	0	0
$x=1$	0.09	0.09	0
$x=2$	0	0	0.81

$$P_{X,Y}(x,y) = \begin{cases} 0.81 & x=2, y=2, \\ 0.09 & x=1, y=1, \\ 0.09 & x=1, y=0, \\ 0.01 & x=0, y=0. \\ 0 & \text{otherwise} \end{cases}$$



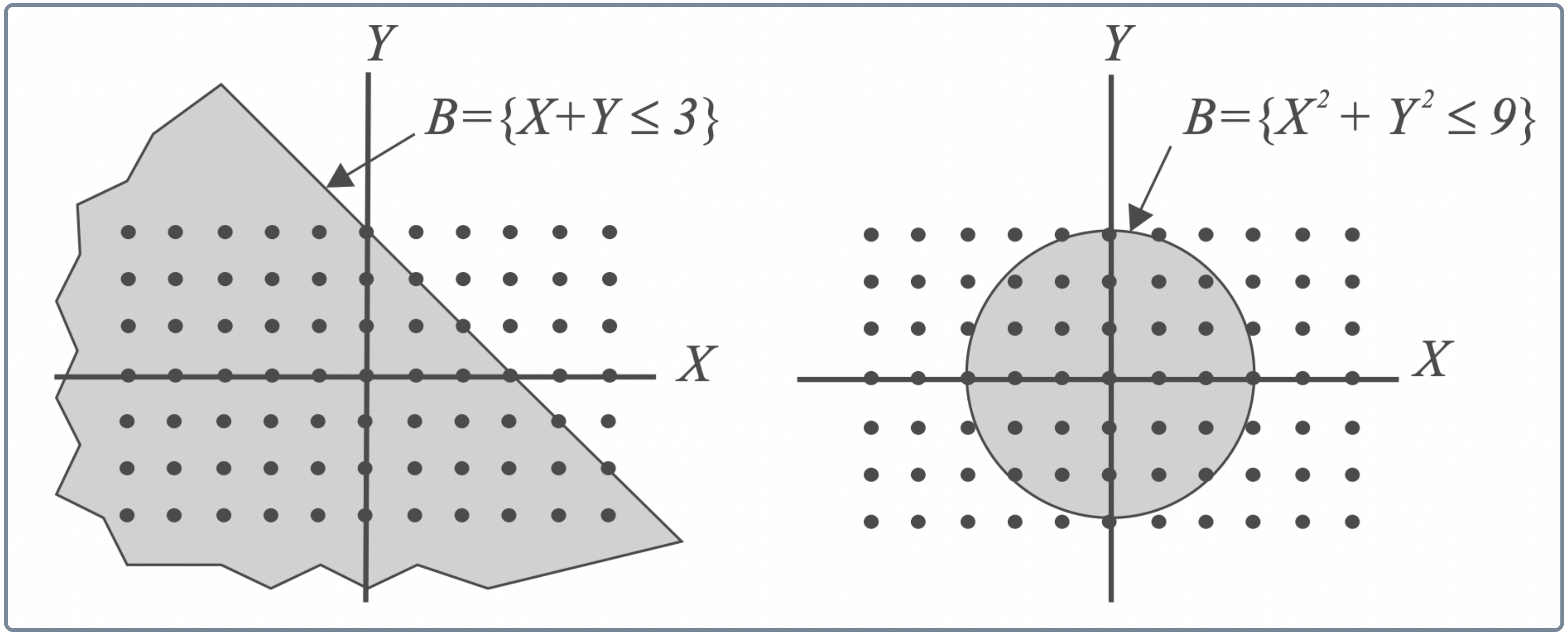


Joint Probability Mass Function

- The range of possible values of the pair (X, Y) is:
  $S_{X,Y} = \{(x, y) | P_{X,Y}(x, y) > 0\}$
- All the probabilities add up to 1.
  $\sum_{x \in S_X} \sum_{y \in S_Y} P_{X,Y}(x, y) = 1$
- Non-negativity
  $P_{X,Y}(x, y) \geq 0$

Joint Probability Mass Function

If we define B as a region in the X, Y plane, e.g.



We can calculate the probability of B as:

$$P(B) = P\left((X, Y) \in B\right) = \sum_{(x,y)\in B} P_{X,Y}(x, y)$$

Marginal PMF

In an experiment that produces two random variables X and Y, it is always possible to consider one of the random variables, Y, and ignore the other one, X.

For discrete random variables X and Y with joint PMF $P_{X,Y}(x, y)$

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x, y)$$

$$P_Y(y) = \sum_{x \in S_X} P_{X,Y}(x, y)$$

In this context, $P_X(x)$ and $P_Y(y)$ are sometimes referred to as the **marginal probability mass functions**.

Example 5.3

Test two integrated circuits one after the other. On each test, the possible outcomes are a (accept) and r (reject). Assume that all circuits are acceptable with probability 0.9 and that the outcomes of successive tests are independent. Count the number of acceptable circuits X and count the number of successful tests Y before you observe the first reject. (If both tests are successful, let $Y = 2$.) Draw a tree diagram for the experiment and find the joint PMF $P_{X,Y}(x, y)$.

- Also find the marginal distributions $P_X(x)$ and $P_Y(y)$.

Example 5.4

Continuing Example 5.3, find the probability of the event B that X, the number of acceptable circuits, equals Y, the number of tests before observing the first failure.

Next lecture

Joint Probability Density Function

The most useful probability model of continuous random variables X and Y is the joint PDF $f_{X,Y}(x,y)$. It is a generalisation of the PDF of a single random variable.

Joint Probability Density Function

The **joint probability density function (PDF)** of the continuous random variables X and Y is the second-order derivative of the joint CDF.

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Also

$$F_{X,Y}(x_1,y_1) = P(X \leq x_1, Y \leq y_1) = \int_{-\infty}^{x_1} \int_{-\infty}^{y_1} f_{X,Y}(x,y)\, dy\, dx$$

Properties of the Joint PDF

1. **Non-negativity**
   $f_{X,Y}(x,y) \geq 0 \;\; \forall \; x, y$
2. **Unity Volume**
   $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx = 1$