Systems and Signals 344

2026

Lecture 13

Delta Functions, Mixed Random Variables

Sections 4.7-4.8

Delta Functions, Mixed Random Variables

X is a mixed random variable if S_X has at least one sample value with nonzero probability (like a discrete random variable) and also has sample values that cover an interval (like a continuous random variable).

The PDF of a mixed random variable contains finite nonzero values and delta functions multiplied by probabilities.

Delta Functions, Mixed Random Variables

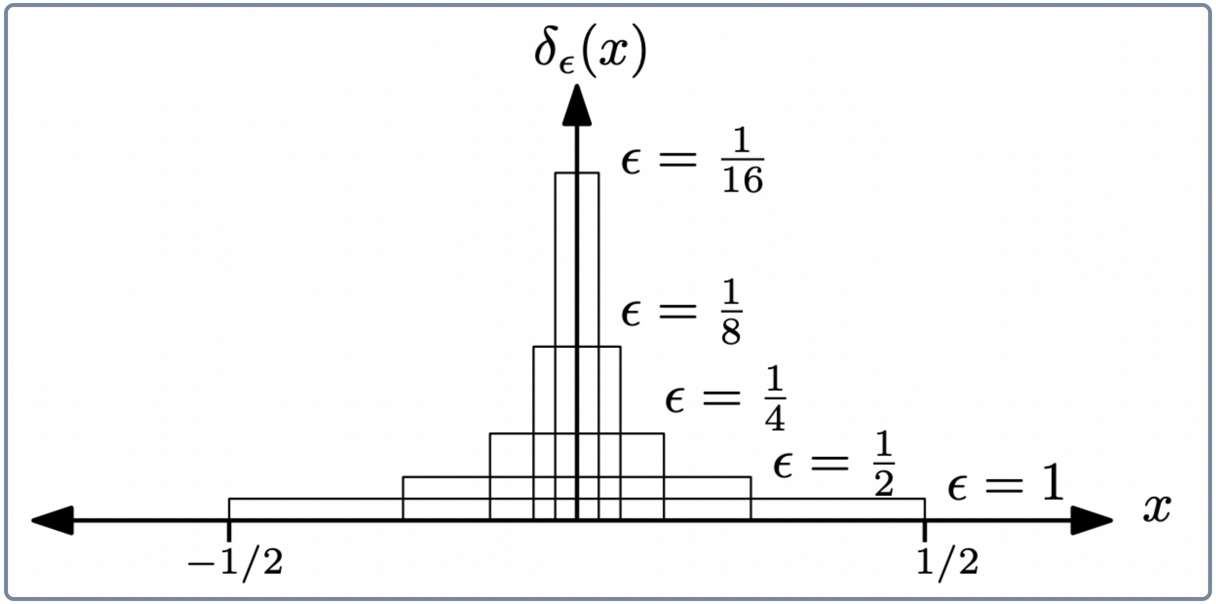
- Due to the different nature of discrete and continuous random variables, we use a PMF for discrete RVs and a PDF for continuous RVs.
- Calculations of probabilities related to a PMF involve sums.
- Calculations of probabilities related to a PDF involve integrals.
- When using a generalised density to indicate probability that exists at a specific value, we use a *unit impulse*, also called a **delta function**.
  - This allows us to use the same formulas to describe both discrete and continuous RVs.
  - We can now *mix* random variables in one formula which contains properties of both continuous and discrete RVs.

Unit Impulse (Delta) Function

$$d_\epsilon(x) = \begin{cases} 1/\epsilon & -\epsilon/2 \le x \le \epsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

The **unit impulse function** is

$$\delta(x) = \lim_{\epsilon \to 0} d_\epsilon(x)$$



Properties of the unit impulse function

The area represented by the unit impulse is always 1.

$$\int_{-\infty}^{\infty} d_\epsilon(x)dx = \int_{-\epsilon/2}^{\epsilon/2} \frac{1}{\epsilon}dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

It exhibits the *Sifting property*

$$\int_{-\infty}^{\infty} g(x)\delta(x - x_0)dx = g(x_0)$$

Unit Step Function

The unit impulse can be used to generate the **unit step function**.

$$u(x) = \int_{-\infty}^{x} \delta(v) dv$$

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

The value $u(0)$ is convention-dependent; here we choose $u(0) = 1$. This is convenient when we discuss discrete random variables again.

Unit Step Function

We also define the inverse as:

$$\delta(x) = \frac{du(x)}{dx}$$

Blurring the lines between discrete RVs and continuous RVs

- The CDF of a discrete random variable, X can be written as: $F_X(x) = \sum_{x_i \in S_X} P_X(x_i)u(x - x_i)$
- To find the generalised density $f_X(x)$, we take the derivative of $F_X(x)$, and therefore the generalised density of a *discrete* random variable X is:

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\delta(x - x_i)$$

We can therefore say that there are simply impulses at x_i

Blurring the lines between discrete RVs and continuous RVs

The expected value can then be calculated as:

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x \sum_{x_i \in S_X} P_X(x_i)\delta(x - x_i)dx$$

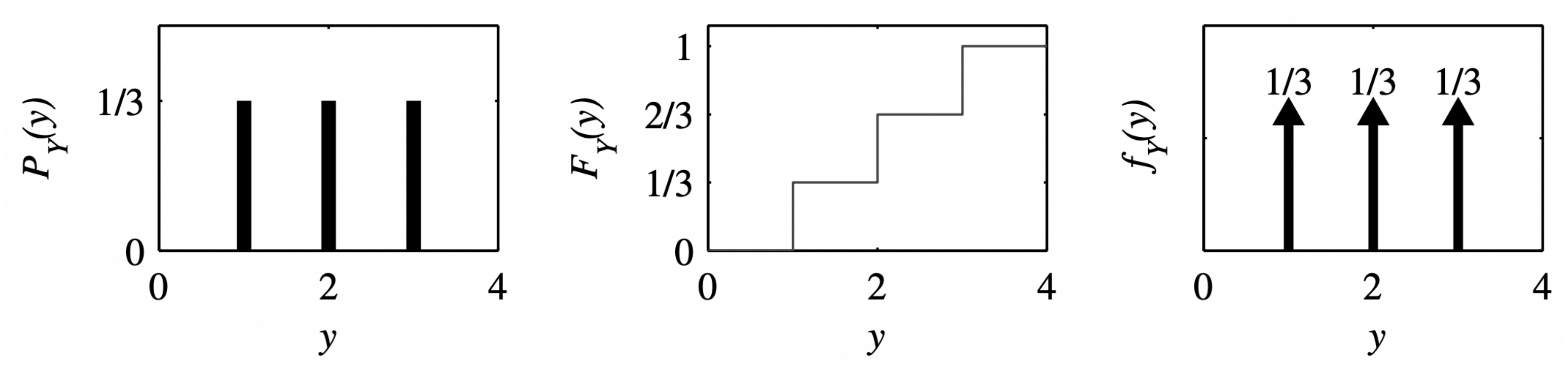
Reorganising and using the *sifting property* for the x

$$\mathrm{E}[X] = \sum_{x_i \in S_X} \int_{-\infty}^{\infty} x P_X(x_i)\delta(x - x_i)dx = \sum_{x_i \in S_X} x_i P_X(x_i)$$

This is exactly what we had for discrete random variables.

Blurring the lines between discrete RVs and continuous RVs

Example



Blurring the lines between discrete RVs and continuous RVs

When $F_X(x)$ has a discontinuity at x, we use $F_X(x^-)$ and $F_X(x^+)$ to denote "from the left" or "from the right".

$$F_X(x^-) = \lim_{h \to 0^+} F_X(x - h)$$

$$F_X(x^+) = \lim_{h \to 0^+} F_X(x + h)$$

The height of the discontinuity, $F_X(x^+) - F_X(x^-)$, indicates the weighting of the impulse in $f_X(x)$.

Blurring the lines between discrete RVs and continuous RVs

Therefore, the following are equivalent (for some arbitrary q):

- $P(X = x_0) = q$
- $P_X(x_0) = q$
- $F_X(x_0^+) - F_X(x_0^-) = q$
- The generalised density contains the term $q\delta(x - x_0)$.

Mixed Random Variable

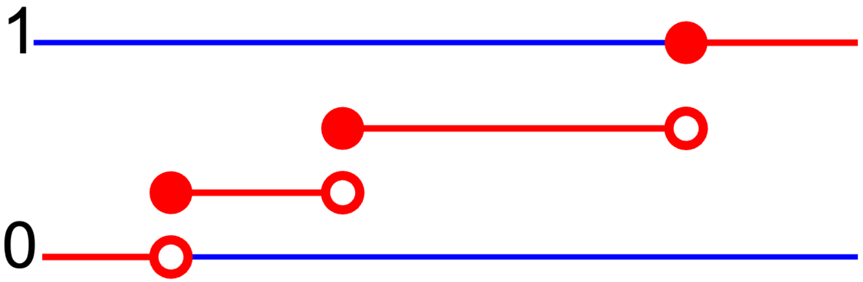
X is a *mixed* random variable if and only if the PDF $f_X(x)$ contains both impulses and non-zero, finite values.

General Random Variables

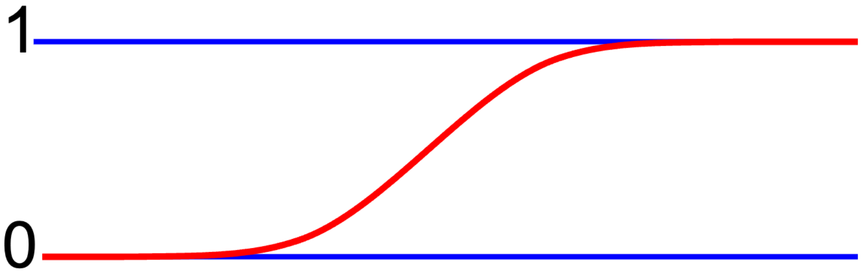
For any random variable X,

- X always has a CDF $F_X(x) = P(X \leq x)$.
- If $F_X(x)$ is piecewise flat with discontinuous jumps, then X is discrete.
- If $F_X(x)$ is a continuous function, then X is continuous.
- If $F_X(x)$ is a piecewise continuous function with discontinuities, then X is mixed.
- When X is discrete or mixed, the PDF $f_X(x)$ contains one or more delta functions.

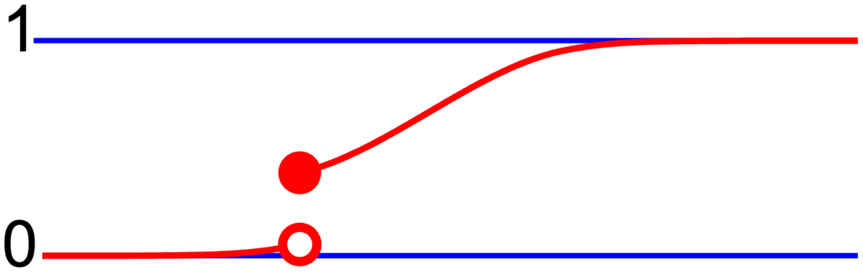
General Random Variables



CDF of Discrete Random Variable



CDF of Continuous Random Variable



CDF of Random Variable with both discrete and continuous parts

Example 4.17

Observe someone dialing a telephone and record the duration of the call. In a simple model of the experiment, $1/3$ of the calls never begin either because no one answers or the line is busy. The duration of these calls is 0 minutes. Otherwise, with probability $2/3$, a call duration is uniformly distributed between 0 and 3 minutes. Let Y denote the call duration. Find the CDF $F_Y(y)$, the PDF $f_Y(y)$, and the expected value $\mathrm{E}[Y]$.

Histograms

- What if we don't know what the density of a random variable is, but we can make a large number of observations?
  - For this we can use histograms, to estimate the density.

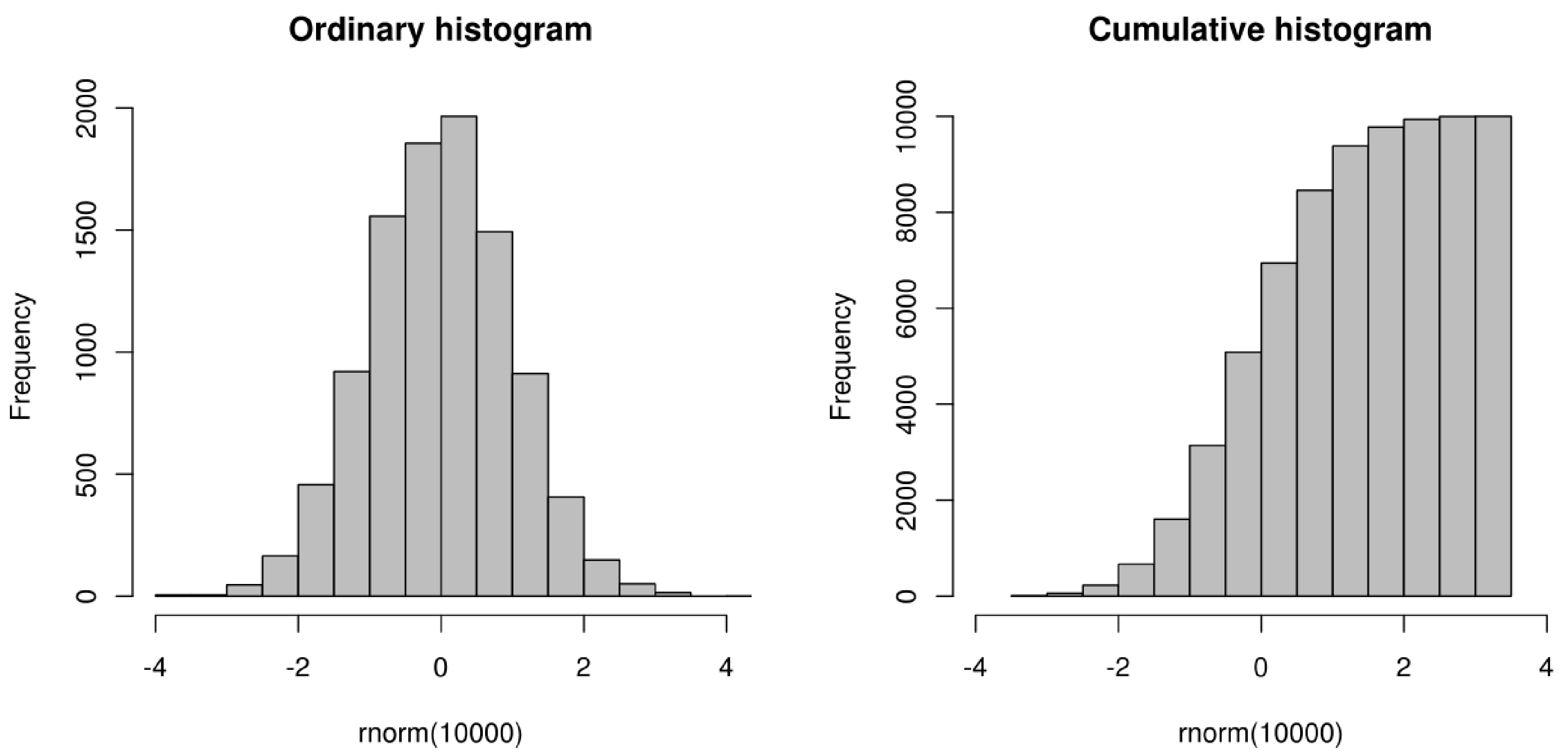
Histograms

- What if we don't know what the probability density function of a random variable is, but we can make a large number of observations?
  - For this we can use histograms, to estimate the density.
- A **histogram** is an approximate representation of the distribution of numerical data.

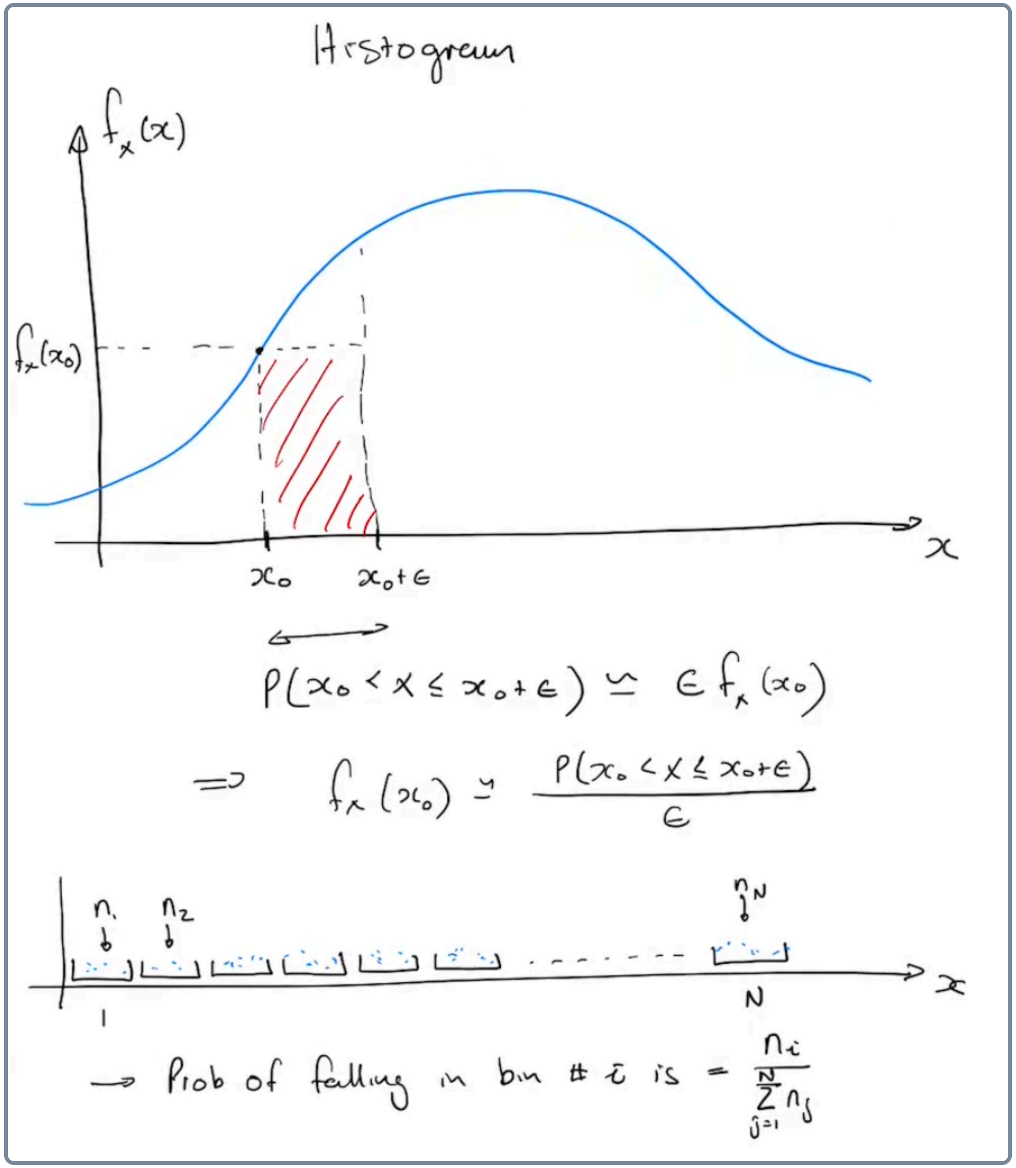
Histograms

- What if we don't know what the probability density function of a random variable is, but we can make a large number of observations?
  - For this we can use histograms, to estimate the density.
- A **histogram** is an approximate representation of the distribution of numerical data.
- A histogram is constructed by first **binning** the range of values into a series of intervals and counting how many values fall into each interval.
  - Bins are usually (but not required) consecutive, non-overlapping, equi-sized intervals of a variable.

Histograms



Ordinary histogram
Frequency
0
500
1000
1500
2000
-4
-2
0
2
4
rnorm(10000)
Cumulative histogram
Frequency
0
2000
4000
6000
8000
10000
-4
-2
0
2
4
rnorm(10000)



Python example