Systems and Signals 344

2026

Lecture 11

Families of Continuous Random Variables

Section 4.5

Recap

The Cumulative Distribution Function

*The **cumulative distribution function** (CDF) of random variable X is:*

$$F_X(x) = P(X \leq x)$$

If X has a PDF, then its CDF $F_X(x)$ is continuous (has no jumps). The converse is not always true: a continuous CDF can be singular and need not have a PDF.

Recap

Probability Density Function

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$F_X(x_1) = \int_{-\infty}^{x_1} f_X(x)dx$$

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx$$

Recap

Probability Density Function

For *continuous random variables*, $P(X = x) = 0$.

Therefore, $f_X(x)$ does not indicate the probability of seeing x, but instead indicates the relative probability located in the region of x.

The *area* between two values, x_1 and x_2 on $f_X(x)$ indicates the probability of seeing X in that range.

Recap

Properties of Probability Density Functions

For a continuous random variable X with PDF $f_X(x)$,

1. **Non-negativity**
   $f_X(x) \geq 0 \ \forall \ x$
2. **Unity Area**
   $\int_{-\infty}^{\infty} f_X(x)dx = 1$

Families of Continuous Random Variables

The families of continuous random variables are related to the families of discrete random variables as follows:

Continuous Uniform $\approx$ ***Discrete Uniform***
Exponential $\approx$ ***Geometric***
Erlang $\approx$ ***Pascal***

Continuous Uniform Distribution

- For a continuous uniform random variable, equal-length subintervals within the range have equal probability.
- This is a natural choice of distribution when the range of values we are trying to model is limited to a finite real interval and we have no reason to prefer any values in this interval above others.
- Hence it often *expresses naivety* about the true value of a quantity.

Continuous Uniform Distribution

X is a **uniform** (a, b) random variable if the PDF of X is

$$f_X(x) = \begin{cases} 1/(b-a) & a \leq x < b \\ 0 & \text{otherwise} \end{cases}$$

where $b > a$. Also note that $S_X = [a, b)$

- If a and b are integers, we can convert from the continuous uniform distribution to the discrete uniform distribution:
  - Make $K = \lceil X \rceil$, then K is the discrete uniform $(a+1, b)$ distribution.
  - Recall that for any x, $\lceil x \rceil$ is the smallest integer greater than or equal to x.

Continuous Uniform Distribution

If X is a **uniform** random variable:

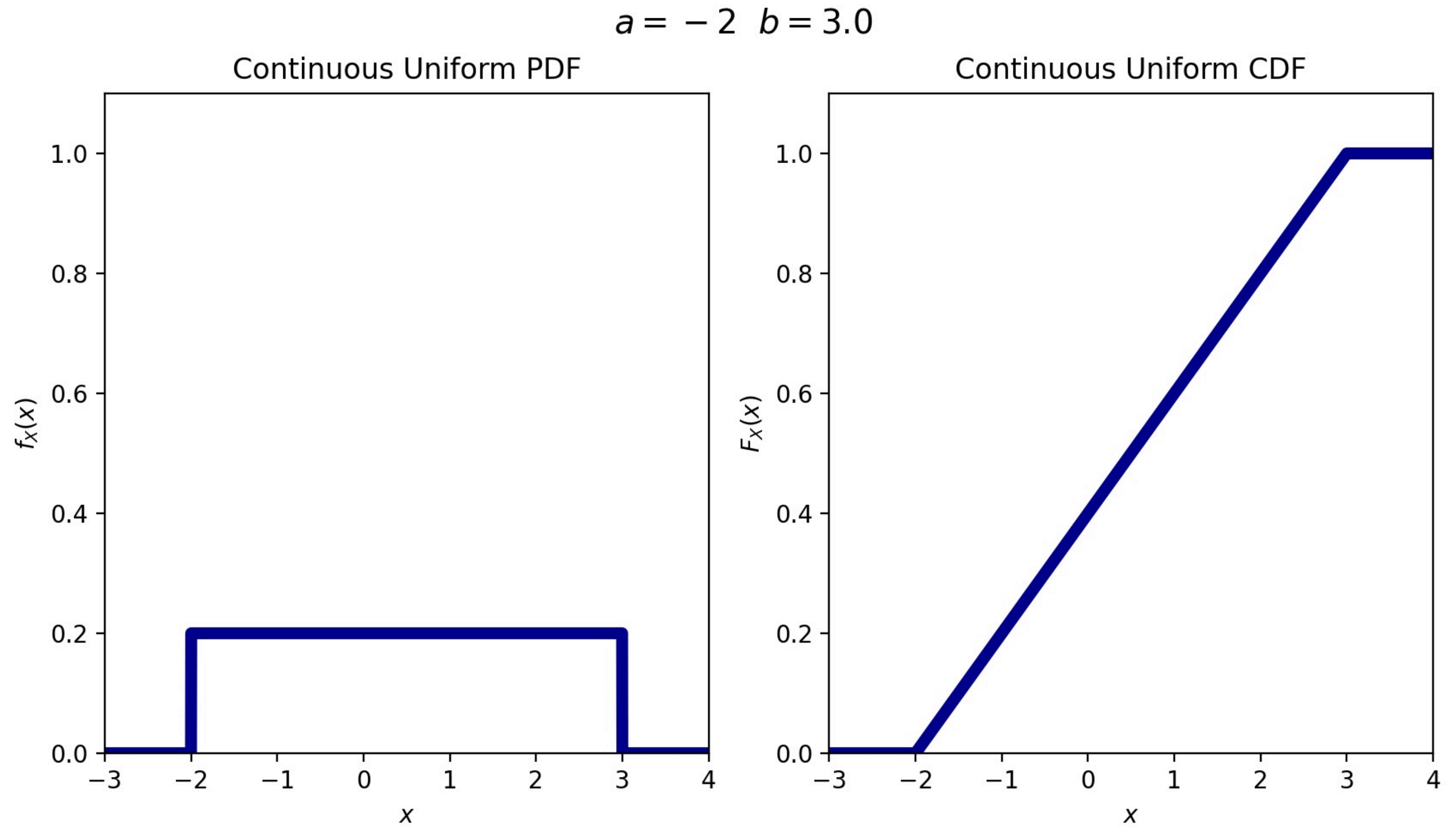
$$F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x > b \end{cases}$$

$$\mathrm{E}[X] = \frac{b+a}{2}$$

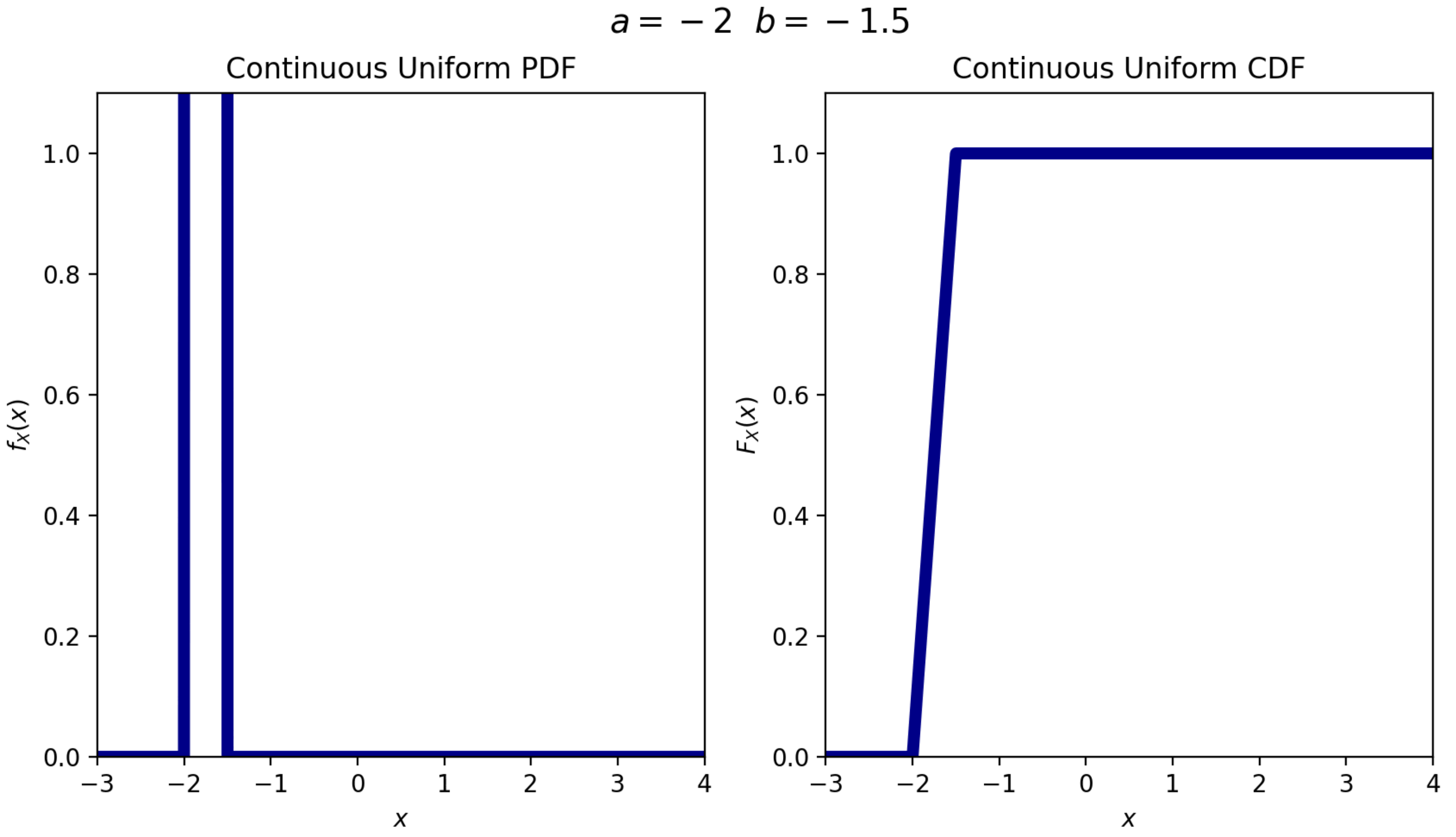
$$\mathrm{Var}[X] = \frac{(b-a)^2}{12}$$

Continuous Uniform Distribution

For $a = -2$ and $b = 3$



Continuous Uniform Distribution



Example 4.9

The phase angle, Θ, of the signal at the input to a modem is uniformly distributed between 0 and 2π radians. What are the PDF, CDF, expected value, and variance of Θ?

Exponential Distribution

- The **exponential distribution** is often used to model the waiting time before a certain event occurs.
- Appropriate when the probability of occurrence of the event increases with the length of the time interval (often a reasonable assumption).
- Also applicable in situations where events occur at a constant rate per unit length:
  - Distance between mutations on a strand of DNA
  - Distance between roadkill on a certain road
- The exponential distribution is the continuous analogue of the *geometric distribution* (for discrete random variables).
  - Make $K = \lceil X \rceil$ and set $p = 1 - e^{-\lambda}$

Exponential Distribution

X is an **exponential** (λ) random variable if the PDF of X is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$.

λ indicates the *rate per unit time*. The mean time between events is $1/\lambda$.
The interpretation of the time unit is determined by the definition of X.

Exponential Distribution

If X is an **exponential** random variable:

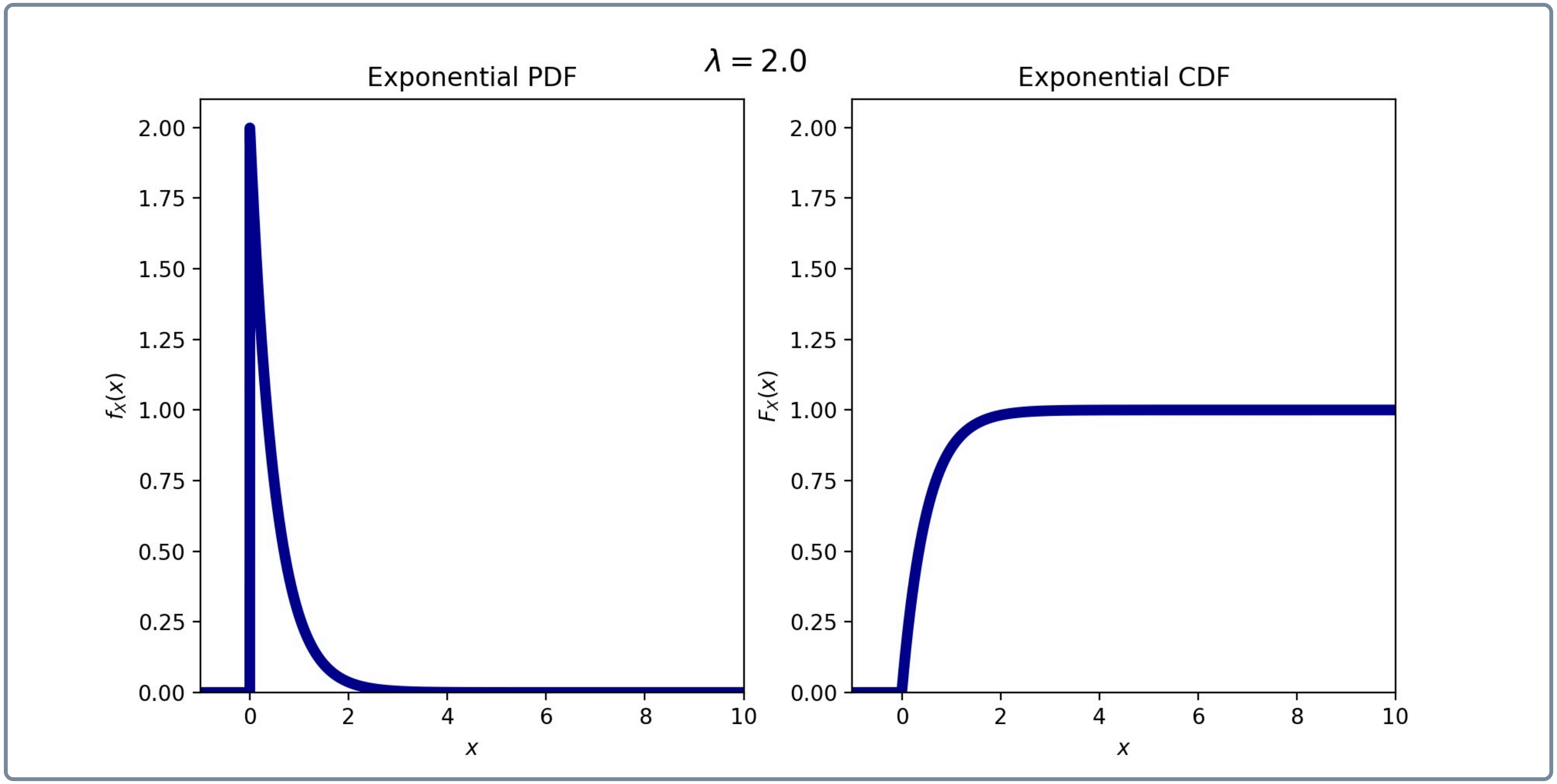
$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mathrm{E}[X] = \frac{1}{\lambda}$$

$$\mathrm{Var}[X] = \frac{1}{\lambda^2} \text{ and } \sigma_X = \frac{1}{\lambda}$$

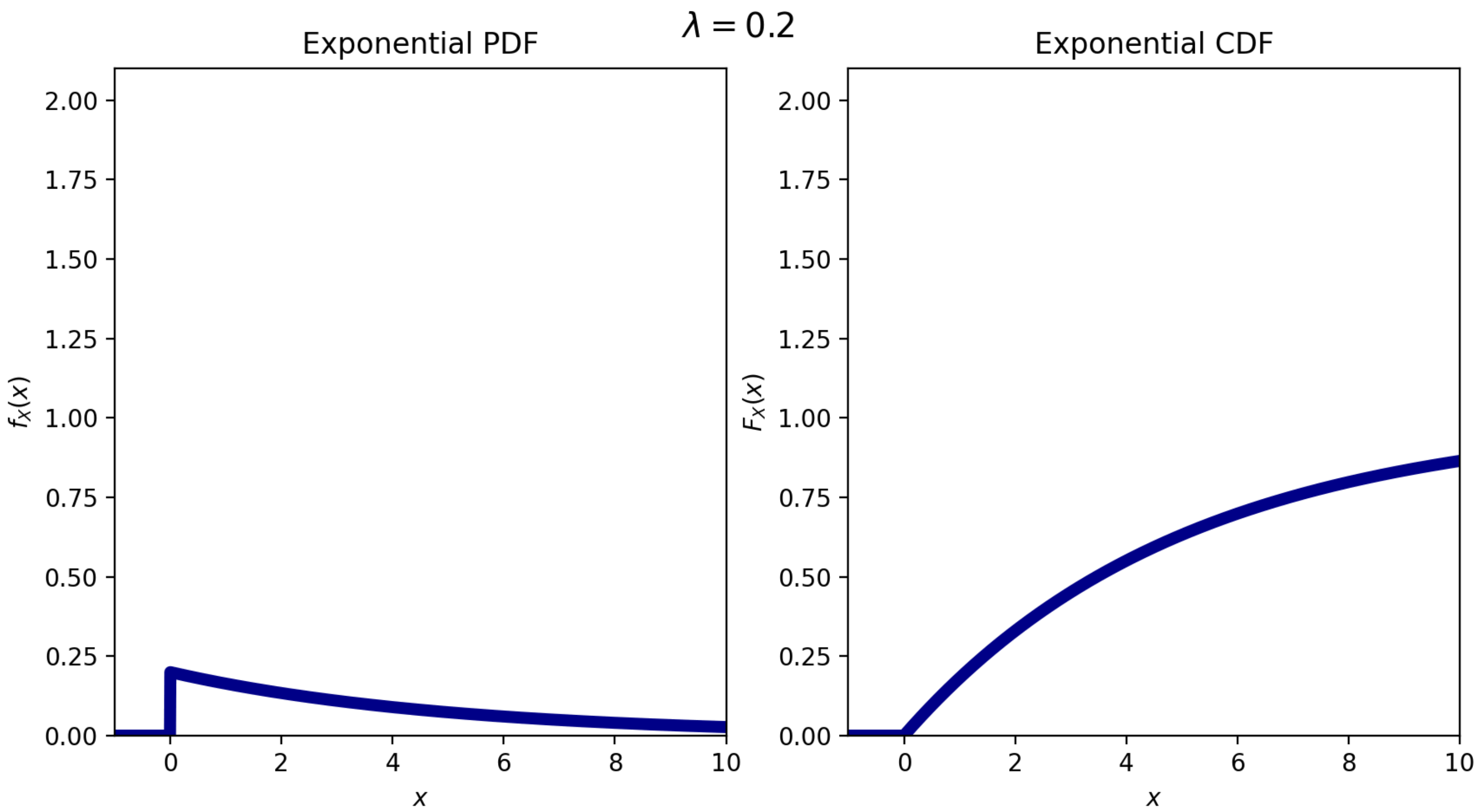
Exponential Distribution

For $\lambda = 2$



Probability greater than 1?

Exponential Distribution



Example 4.10

The duration T of a telephone call is often modeled as an exponential (λ) random variable.. If $\lambda = 1/3$, what is $\mathrm{E}[T]$, the expected duration of a telephone call? What are the variance and standard deviation of T? What is the probability that a call duration is within ± 1 standard deviation of the expected call duration?

T is duration in **minutes**

Erlang Distribution

- The Erlang distribution is the continuous analog of the *Pascal distribution*.
  - Make $K = \lceil X \rceil$ and set $p = 1 - e^{-\lambda}$ and $n = n$.
- Under *continuous* monitoring, the time that we wait for one arrival is an **exponential** (λ) random variable and the time we wait for n arrivals is an **Erlang** (n, λ) random variable.
- On the other hand, when we monitor arrivals in *discrete* one-minute intervals, the number of intervals we wait until we observe a nonempty interval (with one or more arrivals) is a **geometric** $(p = 1 - e^{-\lambda})$ random variable and the number of intervals we wait for n nonempty intervals is a **Pascal** (n, p) random variable.

Erlang Distribution

X is a **Erlang** (n, λ) random variable if the PDF of X is

$$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$ and $n \geq 1$ and n is an integer (known as the *order* of the Erlang random variable).

Note that **Erlang** $(n = 1, \lambda)$ is identical to **exponential** (λ)

Erlang Distribution

If X is a **Erlang** random variable:

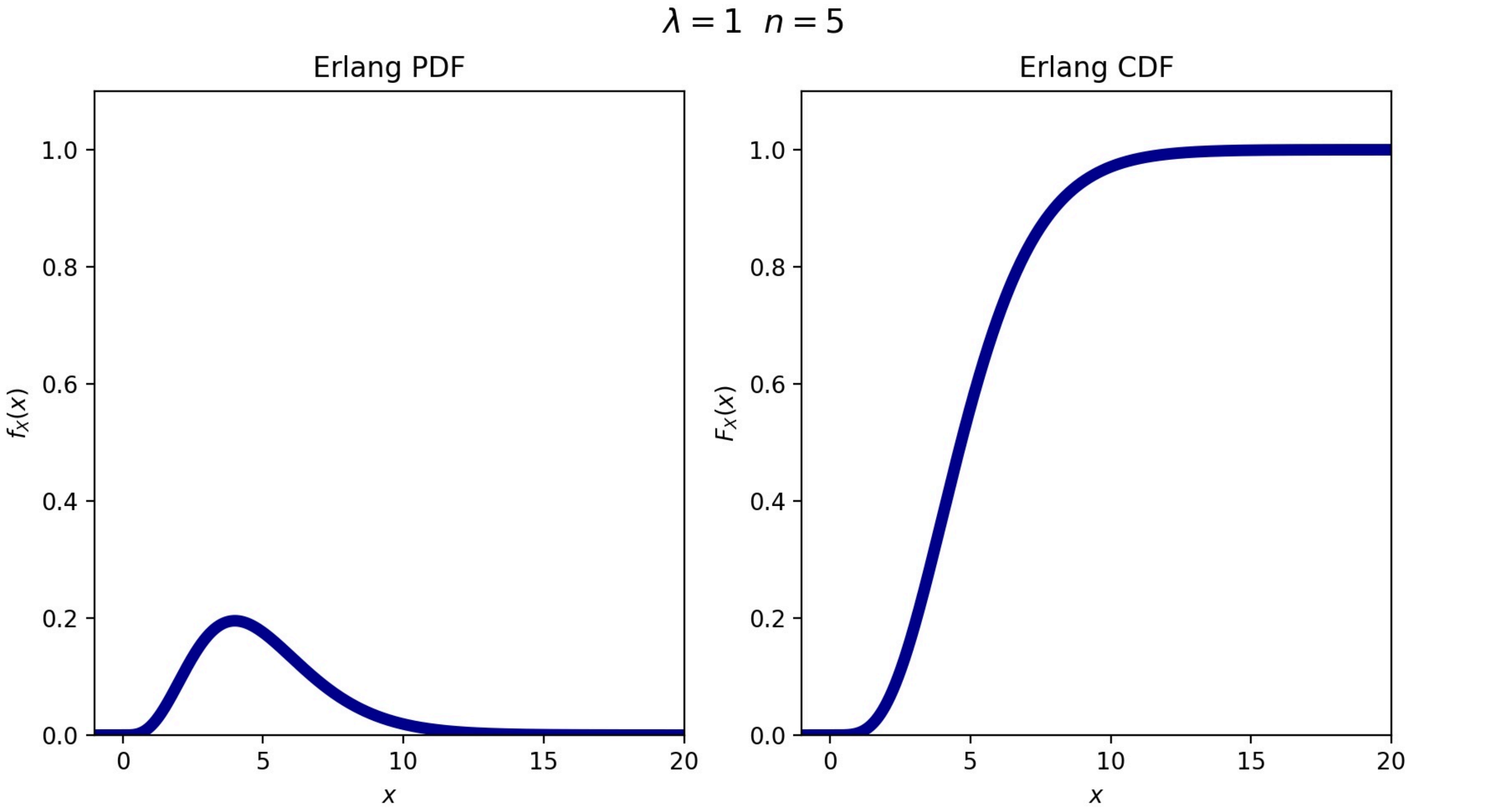
$$F_X(x) = \begin{cases} 1 - \sum_{j=0}^{n-1} \frac{1}{j!} e^{-\lambda x} (\lambda x)^j & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mathrm{E}[X] = \frac{n}{\lambda}$$

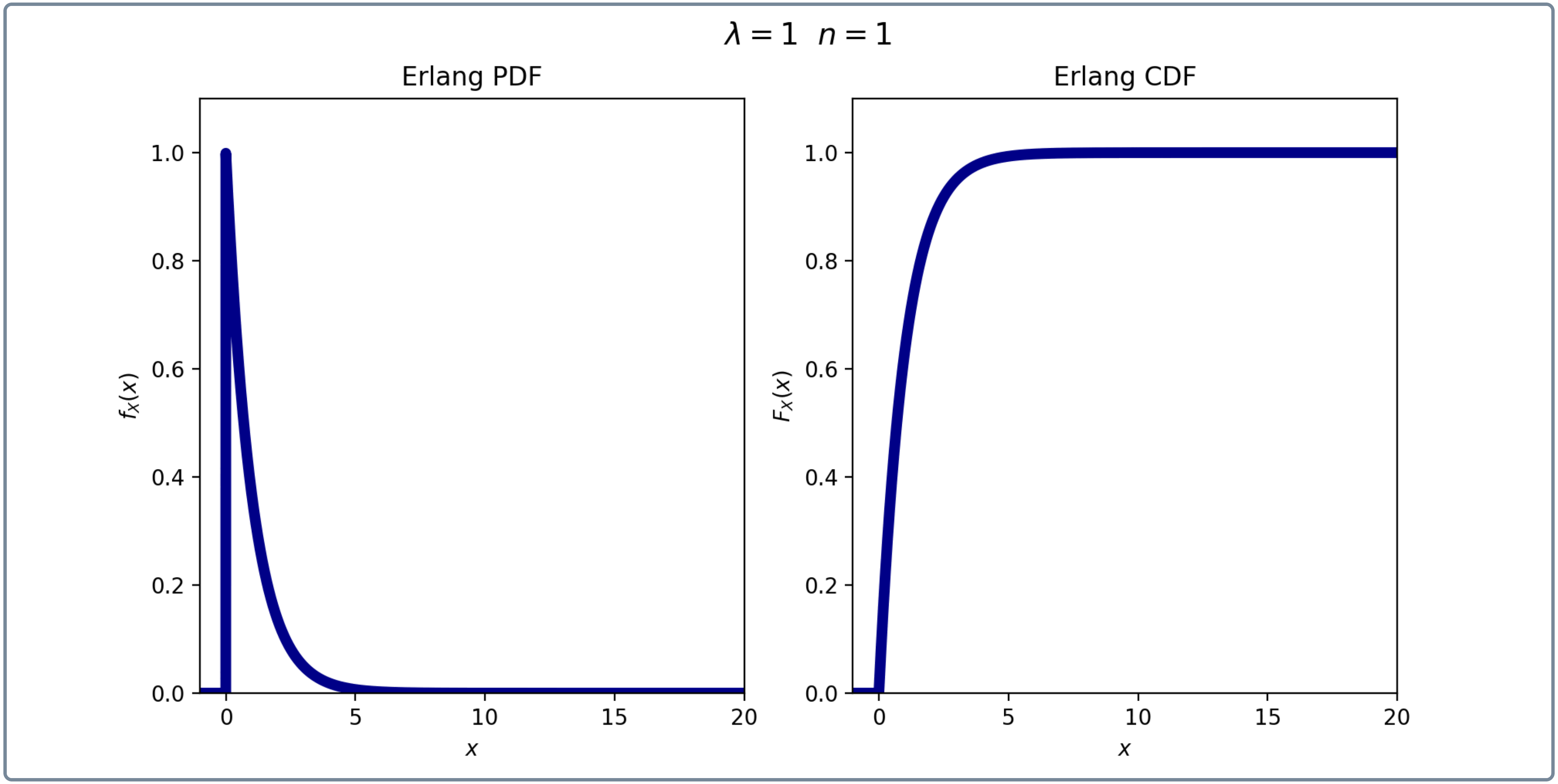
$$\mathrm{Var}[X] = \frac{n}{\lambda^2} \text{ and } \sigma_X = \frac{\sqrt{n}}{\lambda}$$

Erlang Distribution

For $\lambda = 1$ and $n = 5$



Erlang Distribution



Erlang Distribution

The Erlang distribution is also closely related to the Poisson distribution.

Let K_α denote a **Poisson** random variable with parameter α. If X is an **Erlang** random variable with parameters (n, λ), then its CDF is:

$$F_X(x) = \begin{cases} 1 - F_{K_{\lambda x}}(n-1) = 1 - e^{-\lambda x} \sum_{k=0}^{n-1} \dfrac{(\lambda x)^k}{k!}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The mathematical relationships between the **geometric**, **Pascal**, **exponential**, **Erlang**, and **Poisson** distributions derive from the widely used *Poisson process* model.