Systems and Signals 344

2026

Lecture 10

Continuous Sample Space;
The Cumulative Distribution Function;
Probability Density Function; Expected Values

Sections 4.1-4.4

Continuous Sample Space

A random variable X is continuous if the range S_X consists of one or more intervals.

For each $x \in S_X$, $P(X = x) = 0$.

Continuous Sample Space

A **discrete** random variable has a *countable* range of numbers.

A **continuous** random variable has an *uncountably infinite* range of numbers. This is often an *interval* between two limits (e.g. $S_T = \{t \mid 0 \leq t < 10\}$).

The axioms of probability still apply to continuous random variables.

But notably: $P(X = x) = 0$

Example

Modelling the temperature for a specific day.

Continuous Sample Space

But notably: $P(X = x) = 0$

- The more precise your prediction is the lower the probability.
- One way to think about continuous random variables is that the *amount of probability* in an interval gets smaller and smaller as the interval shrinks.
- If X is continuous then $P(X = x) = 0$ for $x \in S_X$, and therefore it is impossible to define a probability mass function $P_X(x)$.
- Probabilities related to continuous random variables are therefore described with **probability density functions** (PDFs) $f_X(x)$.

Example 4.1

Suppose we have a wheel of circumference one meter and we mark a point on the perimeter at the top of the wheel. In the center of the wheel is a radial pointer that we spin. After spinning the pointer, we measure the distance, X meters, around the circumference of the wheel going clockwise from the marked point to the pointer position as shown in Figure 4.1. Clearly, $0 \leq X < 1$. Also, it is reasonable to believe that if the spin is hard enough, the pointer is just as likely to arrive at any part of the circle as at any other. For a given x, what is the probability $\mathrm{P}[X = x]$?

Open the local spinner demo · Online backup

Uniform Distribution

- In many practical applications of probability, we encounter uniform random variables.
- The range of a uniform random variable is an interval with finite limits.
- The probability model of a uniform random variable states that any two intervals of equal size within the range have equal probability.
- To introduce many concepts of continuous random variables, we will refer frequently to a **uniform** random variable with limits 0 and 1. Therefore: $x \in [0, 1)$.
    - Most software libraries that generate random numbers have some kind of `rand` function which produces numbers in this range. For NumPy: `numpy.random.default_rng().random()`

The Cumulative Distribution Function

A **cumulative distribution function** $F_X(x)$ is a very useful *probability model* for continuous random variables.

We can use the CDF to calculate the PDF.

The Cumulative Distribution Function

The ***cumulative distribution function*** *(CDF) of random variable X is:*

$$F_X(x) = P(X \leq x)$$

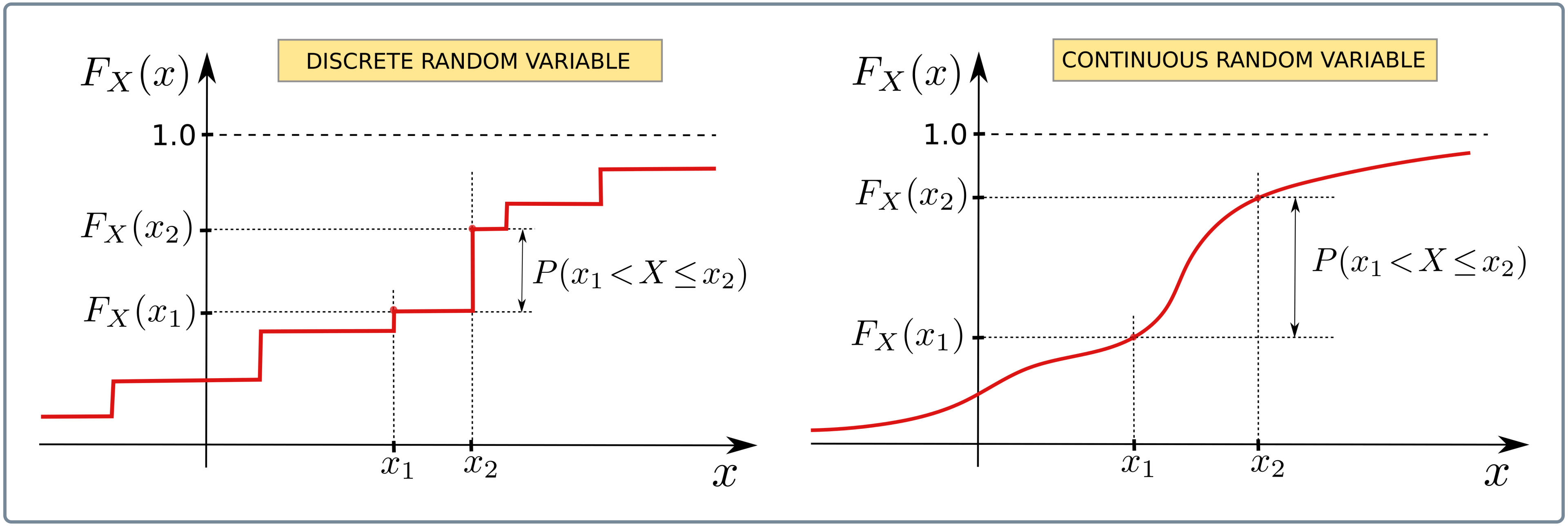
If X has a PDF, then its CDF $F_X(x)$ is continuous (has no jumps). The converse is not always true: a continuous CDF can be singular and need not have a PDF.

In simple terms: The CDF indicates the probability that a random variable X will be *less than or equal* to a specific value x.

Properties of Cumulative Distribution Functions

For any random variable X,

1. $F_X(-\infty) = 0$ and $F_X(\infty) = 1$ and $0 \leq F_X(x) \leq 1$
2. $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) \quad \forall\ x_2 \geq x_1$



Example 4.2

In the wheel-spinning experiment of Example 4.1, find the CDF of X.

Quiz 4.2

The cumulative distribution function of the random variable Y is

$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y/4 & 0 \leq y \leq 4, \\ 1 & y > 4. \end{cases}$$

Sketch the CDF of Y and calculate the following probabilities:

(a) $P[Y \leq -1]$

(b) $P[Y \leq 1]$

(c) $P[2 < Y \leq 3]$

(d) $P[Y > 1.5]$

Probability Density Function

Like the CDF, the PDF $f_X(x)$ *is a probability model for a continuous random variable* X*.*

$f_X(x)$ *is the derivative of the CDF. It is proportional to the probability that* X *is near* x

Probability Density Function

The slope of the CDF contains the most interesting information about a continuous random variable.

The *slope* of the CDF in a region *near* any number x is an indicator of the probability of observing the random variable X *near* x.

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Probability Density Function

$$F_X(x_1) = \int\limits_{-\infty}^{x_1} f_X(x)dx$$

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int\limits_{x_1}^{x_2} f_X(x)dx$$

Note: lower limit x_1 is **not** included in integral, read $\lim_{\epsilon \to 0^+} x_1 + \epsilon$, but since $P(X = x_1) = 0$ it doesn't really matter if the range is *continuous*.

Properties of Probability Density Functions

For a continuous random variable X with PDF $f_X(x)$,

1. **Non-negativity**
   $f_X(x) \geq 0 \ \forall \ x$
2. **Unity Area**
   $\int_{-\infty}^{\infty} f_X(x) dx = 1$

Example 4.3

Figure 4.3 depicts the PDF of a random variable X that describes the voltage at the receiver in a modem. What are probable values of X?

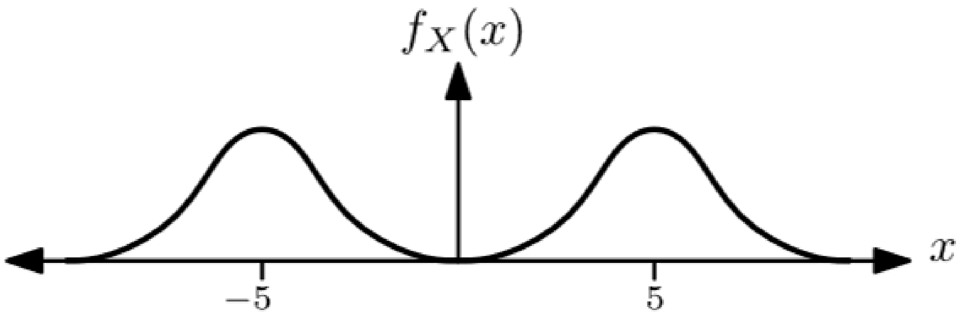


Figure 4.3 The PDF of the modem receiver voltage X.

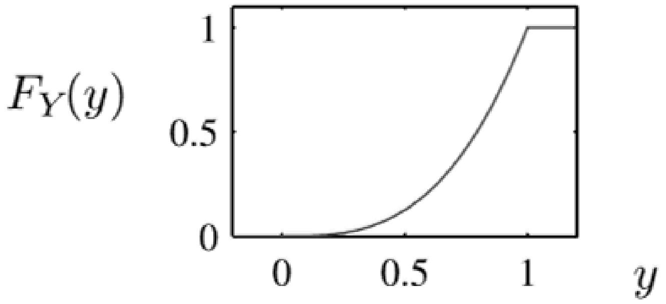
From spinning example:

Example 4.4

For the experiment in Examples 4.1 and 4.2, find the PDF of X and the probability of the event $\{1/4 < X \leq 3/4\}$.

Example 4.5

Consider an experiment that consists of spinning the pointer in Example 4.1 three times and observing Y meters, the maximum value of X in the three spins. In Example 8.3, we show that the CDF of Y is



$$F_Y(y) = \begin{cases} 0 & y < 0, \\ y^3 & 0 \le y \le 1, \\ 1 & y > 1. \end{cases} \tag{4.18}$$

Find the PDF of Y and the probability that Y is between $1/4$ and $3/4$.

Expected Values

Like the expected value of a discrete random variable, the expected value $\mathbf{E}[X]$ *of a continuous random variable* X *is the distribution's mean or centre-of-mass parameter. It is an important property of the probability model of* X.

Expected Values

Provided the integral exists, the **expected value** of a continuous random variable X is

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Expected Values

Provided the relevant integral exists, the **expected value** of a continuous random variable X is

$$\mathrm{E}[X] = \int\limits_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$\mathrm{E}[g(X)] = \int\limits_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx$$

Properties of Expected Values

Same as for discrete random variables, for any random variable X

- $\mathrm{E}[X - \mu_X] = 0$
- $\mathrm{E}[aX + b] = a\mathrm{E}[X] + b$
- $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mu_X^2$
- $\mathrm{Var}[aX + b] = a^2\,\mathrm{Var}[X]$

Properties of Expected Values

For any continuous random variable X:

$$\mathrm{E}[X^2] = \int\limits_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx$$

$$\mathrm{Var}[X] = \int\limits_{-\infty}^{\infty} (x - \mu_X)^2 \cdot f_X(x)\, dx$$

Value	Discrete	Continuous
$\mathrm{E}[X]$	$\sum_{x \in S_X} x P_X(x)$	$\int_{-\infty}^{\infty} x f_X(x)\, dx$
$\mathrm{Var}[X]$	$\sum_{x \in S_X} (x - \mu_X)^2 P_X(x)$	$\int_{-\infty}^{\infty} (x - \mu_X)^2 \cdot f_X(x)\, dx$