Systems and Signals 344

2026

Lecture 9

Families of Discrete Random Variables (continued)

Sections 3.3-3.9

Recap

Bernoulli Trials

- Consider an experiment with *independent trials* in which each subexperiment has *two* possible outcomes
  - A (success) with $P(A) = p$
  - $\bar{A}$ (failure) with $P(\bar{A}) = 1 - p$
- These subexperiments are called **Bernoulli trials**.
  - Depending on how the outcomes of the random variable X are defined we call these models by different names, such as **Bernoulli**, **binomial**, **geometric**, and **Pascal** distributions.

Bernoulli Trials

- **Bernoulli Distribution (p):** Conducting a single Bernoulli Trial
- **Binomial Distribution (n, p):** Probability of observing exactly x successes in n trials
- **Geometric Distribution (p):** Probability of requiring x trials until seeing the first success
- **Pascal Distribution (k, p):** Probability of observing exactly k successes in x trials.
    - Distribution of the trial number X on which the kth success occurs.
    - Note if $k = 1$ then this is the same as the Geometric Distribution.

Recap

Binomial Distribution

When conducting n **Bernoulli trials**, if we want to know the probability that A is observed *exactly* x times, then we call the resulting distribution a **Binomial distribution**.

X is a **binomial** (n, p) random variable if the PMF of X has the form

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $0 < p < 1$ and n is an integer such that $n \geq 1$ and $0 \leq x \leq n$.

Pascal Distribution

If we want to know the probability that A is observed *exactly* k times in x **Bernoulli trials**, then we call the resulting distribution a **Pascal distribution**.

X is a **Pascal** (k, p) random variable if the PMF of X has the form

$$P_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

where $0 < p < 1$ and k is an integer such that $k \geq 1$.
Note that it is only non-zero for $x = k, k+1, \ldots$

Note that the **Pascal distribution** is a special case of the **Negative binomial distribution** (which we don't cover).

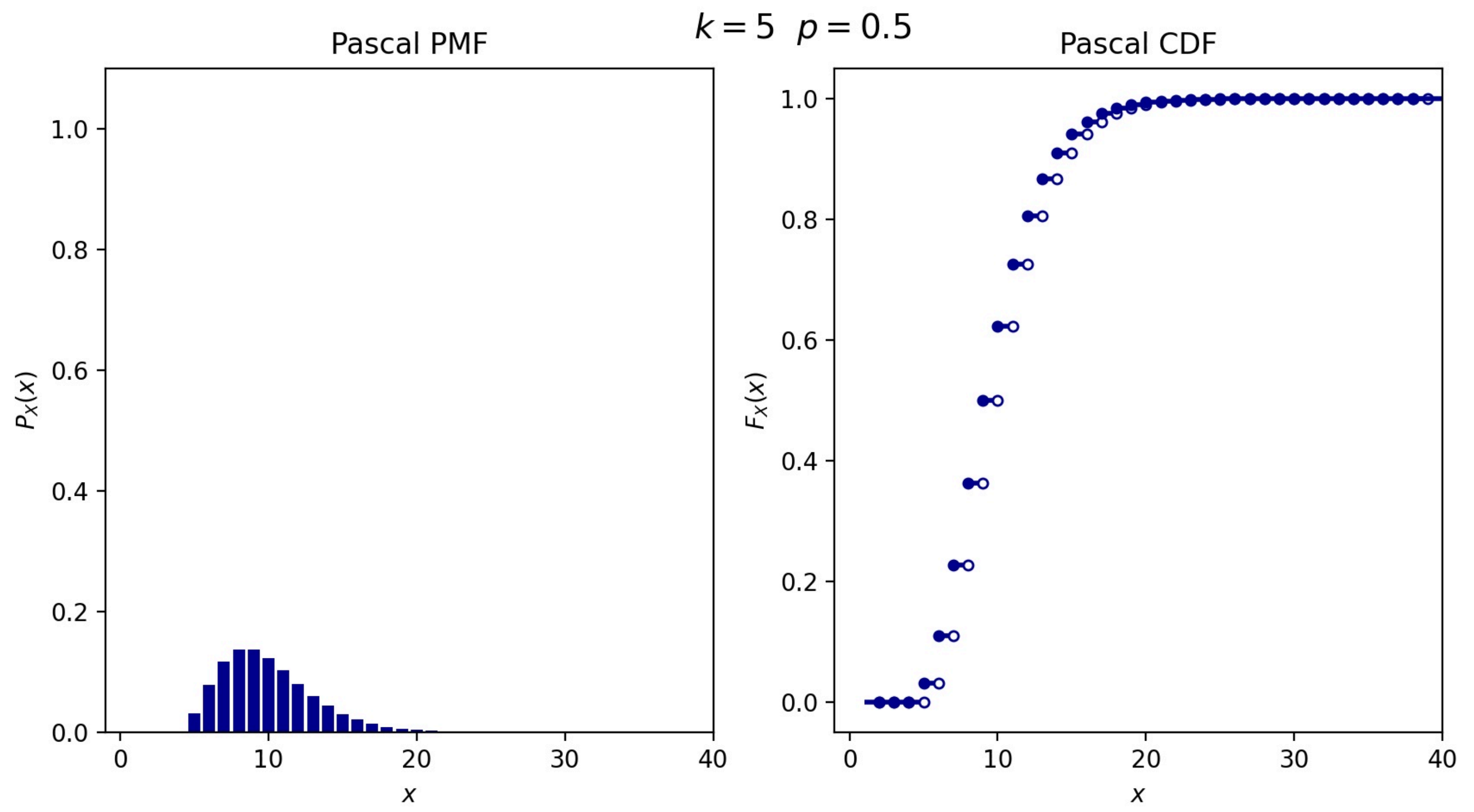
Pascal Distribution

The **Pascal** CDF is defined as

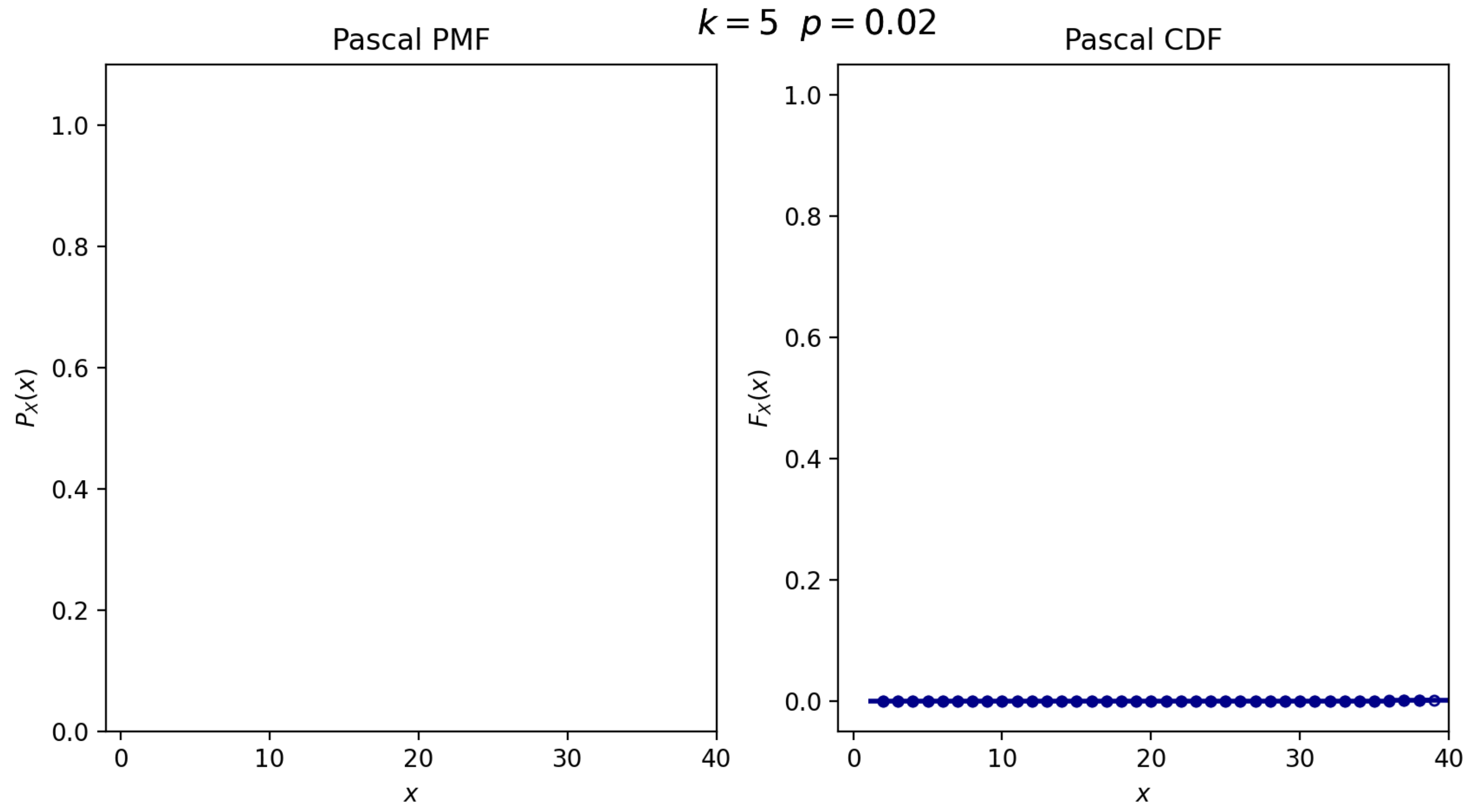
$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= \begin{cases} 0, & x < k, \\ \displaystyle\sum_{n=k}^{\lfloor x \rfloor} \binom{n-1}{k-1} p^k (1-p)^{n-k}, & x \geq k. \end{cases}
\end{aligned}
$$

Pascal Distribution

For $k = 5$ and $p = 0.5$



Pascal Distribution



Pascal Distribution

Expected Value

$$\mathrm{E}[X] = \frac{k}{p}$$

$$\mathrm{E}[X^2] = \frac{k+k(1-p)}{p^2}$$

Variance

$$\mathrm{Var}[X] = \frac{k(1-p)}{p^2}$$

Example 3.10

Perform independent tests of integrated circuits in which each circuit is rejected with probability p. Observe L, the number of tests performed until there are k rejects. What is the PMF of L?

Discrete Uniform Distribution

In an experiment with *equiprobable outcomes*, random variable X has the range $S_X = \{k, k+1, k+2, \ldots, l\}$. The range contains $n = l - k + 1$ numbers, each with a probability of $1/n$.

This is a natural choice of distribution when the values we are trying to model are limited to a finite set of consecutive integers and we have no reason to prefer any value above another.

Hence it often *expresses naivety* about the true value of a quantity.

Discrete Uniform Distribution

X is a **discrete uniform** (k, l) random variable if the PMF of X has the form (for $x \in [k, l]$)

$$P_X(x) = \begin{cases} 1/(l-k+1) & x = k, k+1, k+2, \ldots, l \\ 0 & \text{otherwise} \end{cases}$$

where the parameters k and l are integers such that $k < l$.

We often say "X is uniformly distributed between k and l."

Discrete Uniform Distribution

The **uniform distribution** CDF is defined as

$$F_X(x) = \begin{cases} 0 & x < k \\ \frac{x-k+1}{l-k+1} & x = k, k+1, k+2, \ldots, l \\ 1 & x > l \end{cases}$$

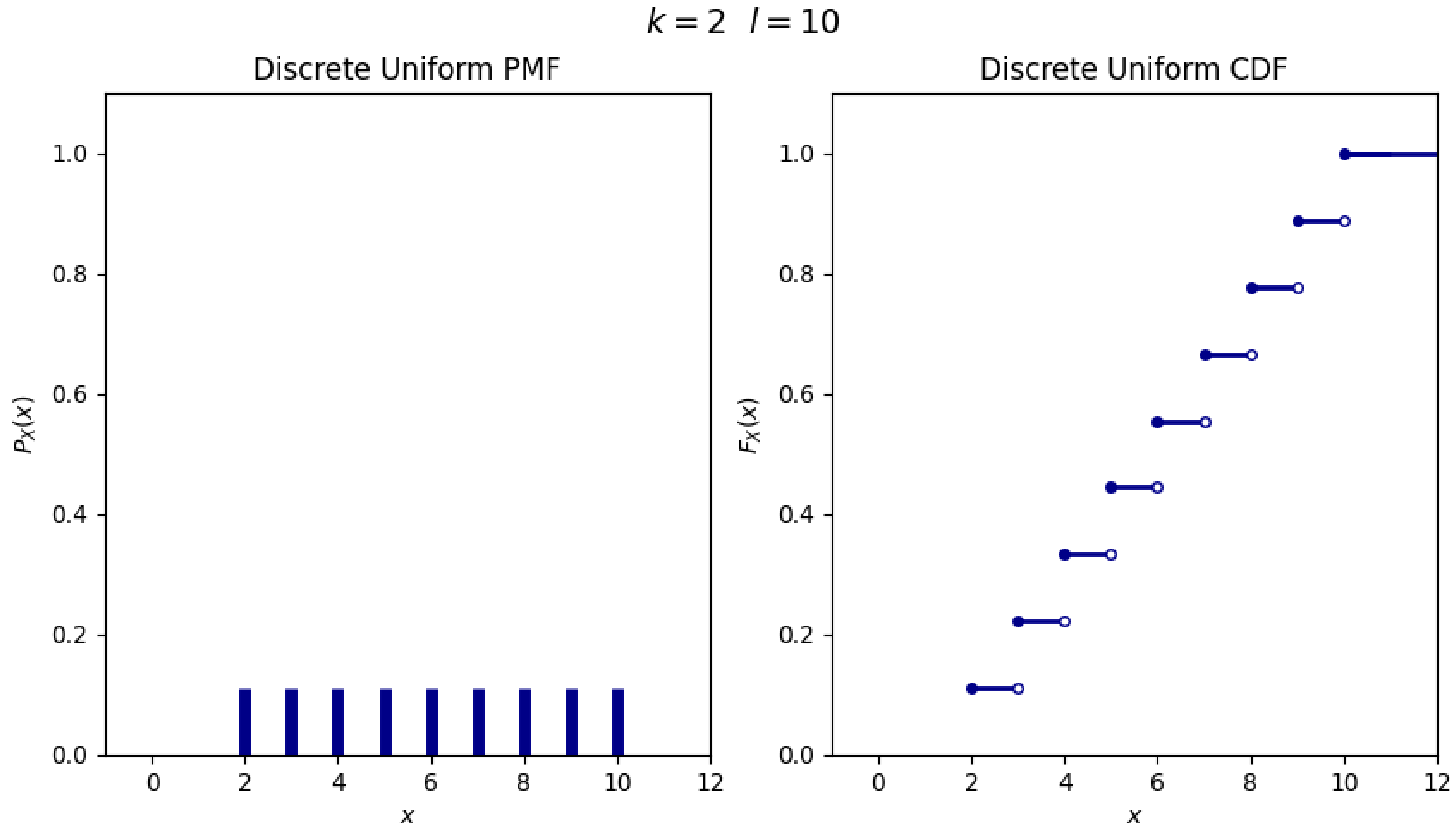
Discrete Uniform Distribution

The **uniform distribution** CDF is defined as

$$F_X(x) = \begin{cases} 0 & x < k \\ \frac{\lfloor x \rfloor - k + 1}{l - k + 1} & k \leq x < l \\ 1 & x \geq l \end{cases}$$

Discrete Uniform Distribution

For $k = 2$ and $l = 10$



Discrete Uniform Distribution

Expected Value

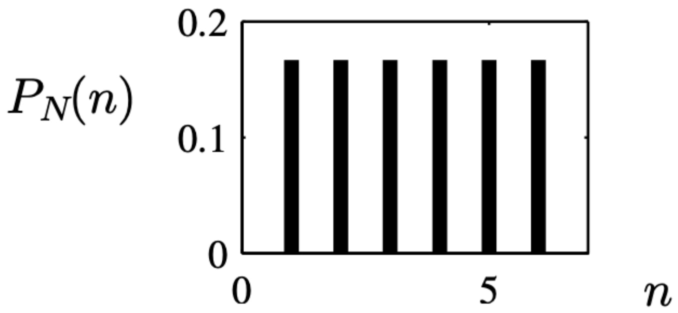
$$\mathrm{E}[X] = \frac{k+l}{2}$$

Variance

$$\mathrm{Var}[X] = \frac{(l-k)(l-k+2)}{12}$$

Example 3.12

Roll a fair die. The random variable N is the number of spots on the side facing up. Therefore, N is a discrete uniform $(1,6)$ random variable with PMF



$$P_N(n) = \begin{cases} 1/6 & n = 1,2,3,4,5,6, \\ 0 & \text{otherwise.} \end{cases} \tag{3.19}$$

Poisson Distribution

- Consider arrivals generated by a Poisson process: arrivals occur independently, the *rate is constant*, and the probability of more than one arrival in a sufficiently short interval is negligible.
- We call the occurrence of the phenomenon of interest the *arrival*.
- The Poisson model often specifies the average rate λ arrivals per second, and a *time interval*, T seconds.
- In this time interval T, the number of arrivals X has a Poisson PMF with $\alpha = \lambda T$ (note that α is dimensionless)

Poisson Distribution

X is a **Poisson** (α) random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where the parameter α is in the range $\alpha > 0$.

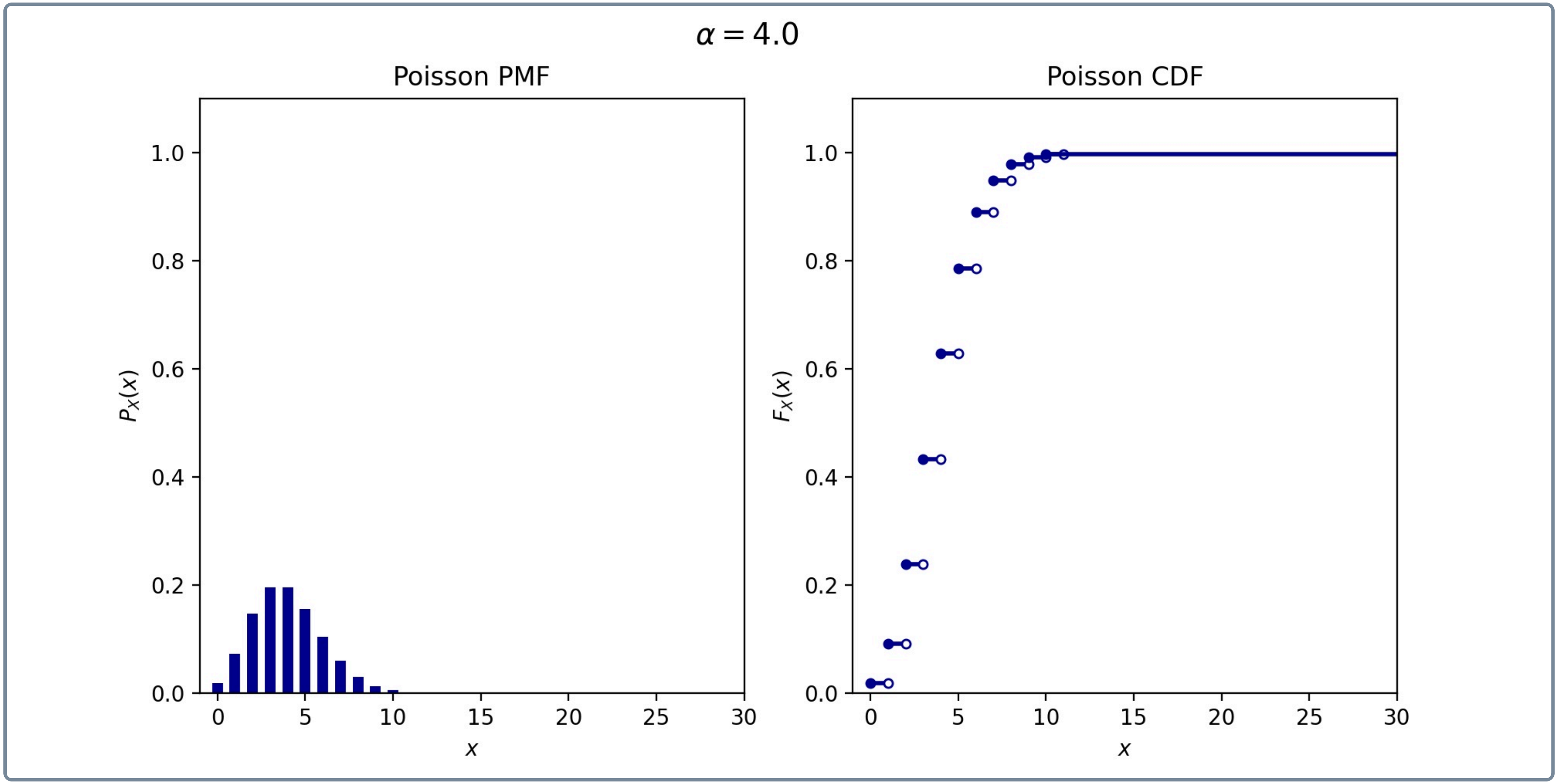
Poisson Distribution

The **Poisson** CDF is defined as

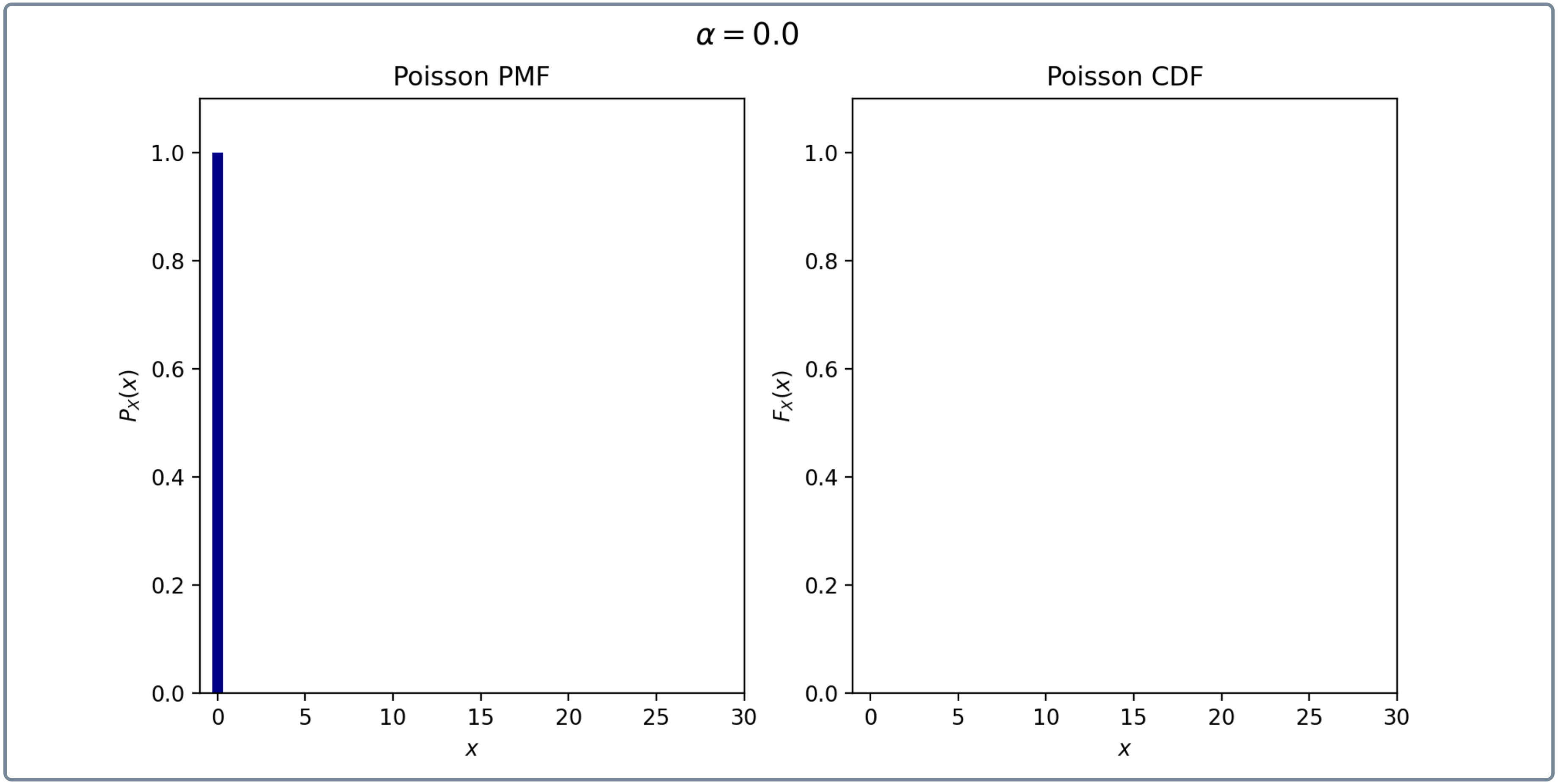
$$F_X(x) = \begin{cases} 0, & x < 0, \\ e^{-\alpha} \displaystyle\sum_{i=0}^{\lfloor x \rfloor} \frac{\alpha^i}{i!}, & x \geq 0. \end{cases}$$

Poisson Distribution

For $\alpha = 4$



Poisson Distribution



Poisson Distribution

Expected Value

$$\mathrm{E}[X] = \alpha$$

Variance

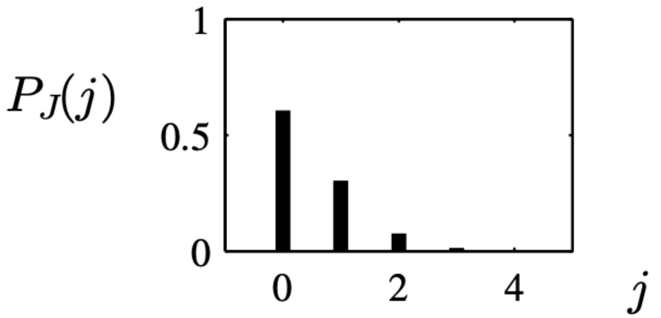
$$\mathrm{Var}[X] = \alpha$$

Example 3.13

The number of hits at a website in any time interval is a Poisson random variable. A particular site has on average $\lambda = 2$ hits per second. What is the probability that there are no hits in an interval of 0.25 seconds? What is the probability that there are no more than two hits in an interval of one second?

Example 3.14

Calls arrive at random times at a telephone switching office with an average of $\lambda = 0.25$ calls/second. The PMF of the number of calls that arrive in a $T = 2$-second interval is the Poisson (0.5) random variable with PMF



$$P_J(j) = \begin{cases} (0.5)^j e^{-0.5}/j! & j = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Note that we obtain the same PMF if we define the arrival rate as $\lambda = 60 \cdot 0.25 = 15$ calls per minute and derive the PMF of the number of calls that arrive in $2/60 = 1/30$ minutes.