Systems and Signals 344

2026

Lecture 8

Families of Discrete Random Variables

Sections 3.3-3.9

Families of Discrete Random Variables

In applications of probability, many experiments have similar probability mass functions (PMFs). In a family of random variables, the PMFs of the random variables have the same mathematical form, differing only in the values of one or two parameters.

See Appendix A for a summary of all the relevant ones.

Bernoulli Trials

- Consider an experiment with *independent trials* in which each subexperiment has *two* possible outcomes
  - A (success) with $P(A) = p$
  - $\bar{A}$ (failure) with $P(\bar{A}) = 1 - p$
- These subexperiments are called **Bernoulli trials**.
  - Depending on how the outcomes of the random variable X are defined we call these models by different names, such as **Bernoulli**, **binomial**, **geometric**, and **Pascal** distributions.

Bernoulli Distribution

A *Bernoulli Trial* in which the random variable (X) takes the value 1 for success and 0 for failure (i.e. $S_X = \{0, 1\}$) is called a **Bernoulli random variable**.

NOTE: it is common to use Random Variable K (instead of X) when discussing discrete distributions

Bernoulli Distribution

A *Bernoulli Trial* in which the random variable (X) takes the value 1 for success and 0 for failure (i.e. $S_X = \{0, 1\}$) is called a **Bernoulli random variable**.

X is a **Bernoulli** (p) random variable if the PMF of X has the form:

$$P_X(x) = \begin{cases} 1 - p & x = 0 \\ p & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

where the parameter p is in the range $0 < p < 1$

Bernoulli Distribution

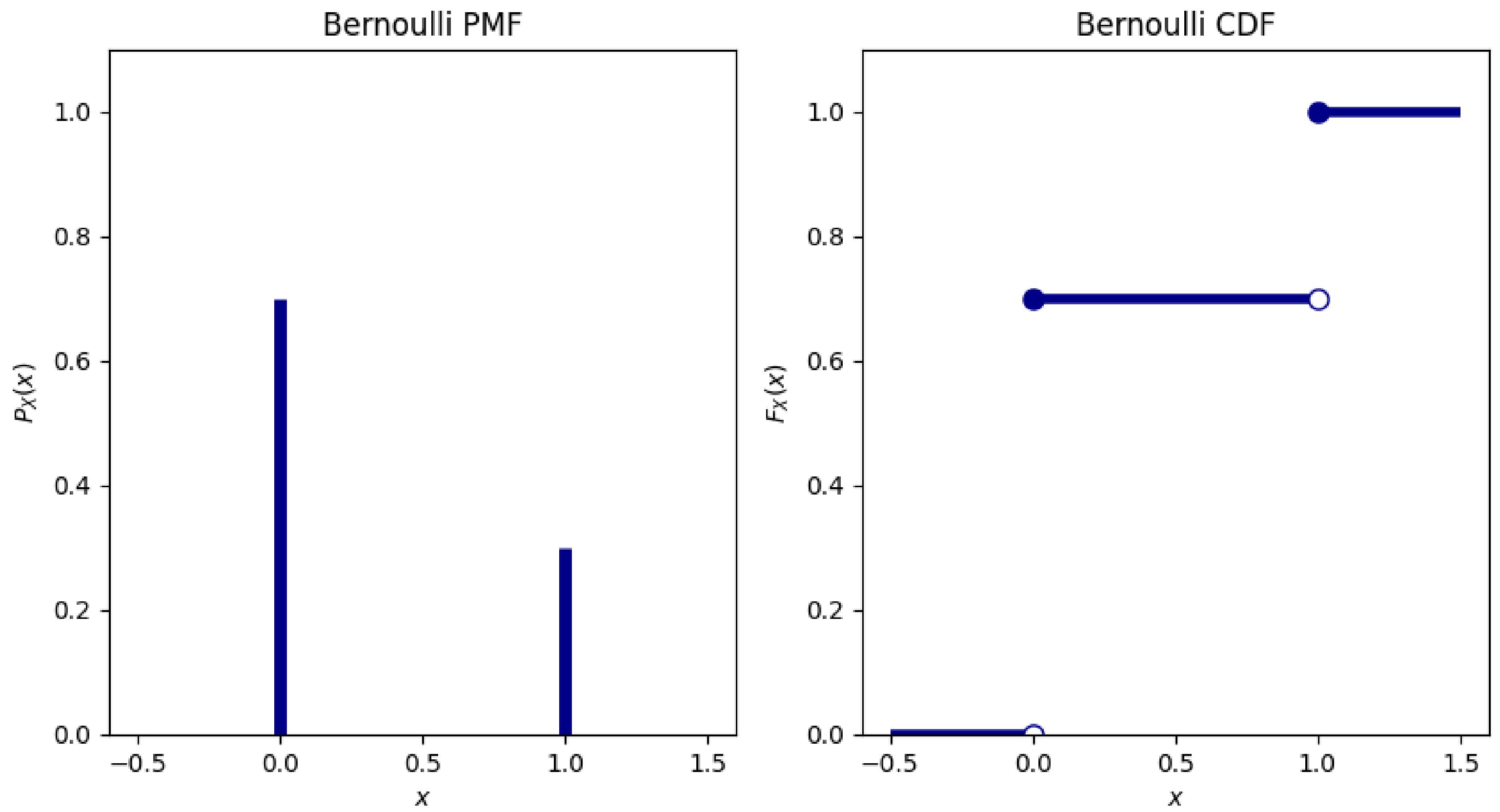
The **Bernoulli** CDF is defined as

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < 0 \\ 1 - p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Don't let the "continuous" x confuse you, it still has discrete jumps at $x = 0$ and $x = 1$.

Bernoulli Distribution

For $p = 0.3$



Bernoulli Distribution - Expected Value

$$
\begin{aligned}
\mathrm{E}[X] = \mu_X = \sum_{x=0}^{1} x \cdot P_X(x) \\
&= 0 \cdot P_X(0) + 1 \cdot P_X(1) \\
&= 0 \cdot (1-p) + 1 \cdot (p) \\
&= p
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[X^2] = \sum_{x=0}^{1} x \cdot P_X(x) \\
&= 0 \cdot P_X(0) + 1 \cdot P_X(1) \\
&= 0 \cdot (1-p) + 1 \cdot (p) \\
&= p
\end{aligned}
$$

Bernoulli Distribution

Variance

$$\begin{aligned}\mathrm{Var}[X] &= \mathrm{E}[X^2] - \mathrm{E}[X]^2 \\ &= p - p^2 \\ &= p(1-p)\end{aligned}$$

Variance is maximum at $p = 0.5$

Bernoulli Trials

- **Bernoulli Distribution (p):** Conducting a single Bernoulli Trial
- **Binomial Distribution (n, p):**
- **Geometric Distribution (p):**
- **Pascal Distribution (k, p):**

Binomial Distribution

- When conducting n **Bernoulli trials** (repetitions), if we want to know the probability that A is observed *exactly* x times, then we call the resulting distribution a **Binomial distribution**.
- The probability of observing any specific sequence of n outcomes in which A occurs x times and therefore $\bar{A}$ occurs $(n-x)$ times is:

$$\prod_{x} p \prod_{n-x} (1-p) = p^x (1-p)^{n-x}$$

- But there are $\binom{n}{x}$ possible sequences of length n in which A occurs x times, each with probability $p^x(1-p)^{n-x}$. Therefore...

Example 3.9

In a sequence of n independent tests of integrated circuits, each circuit is rejected with probability p. Let K equal the number of rejects in the n tests. Find the PMF $P_K(k)$.

Binomial Distribution

When conducting n **Bernoulli trials**, if we want to know the probability that A is observed *exactly* x times, then we call the resulting distribution a **Binomial distribution**.

X is a **binomial** (n, p) random variable if the PMF of X has the form

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $0 < p < 1$ and n is an integer such that $n \geq 1$ and $0 \leq x \leq n$.

Note: For $n = 1$ we simply have the Bernoulli Distribution.

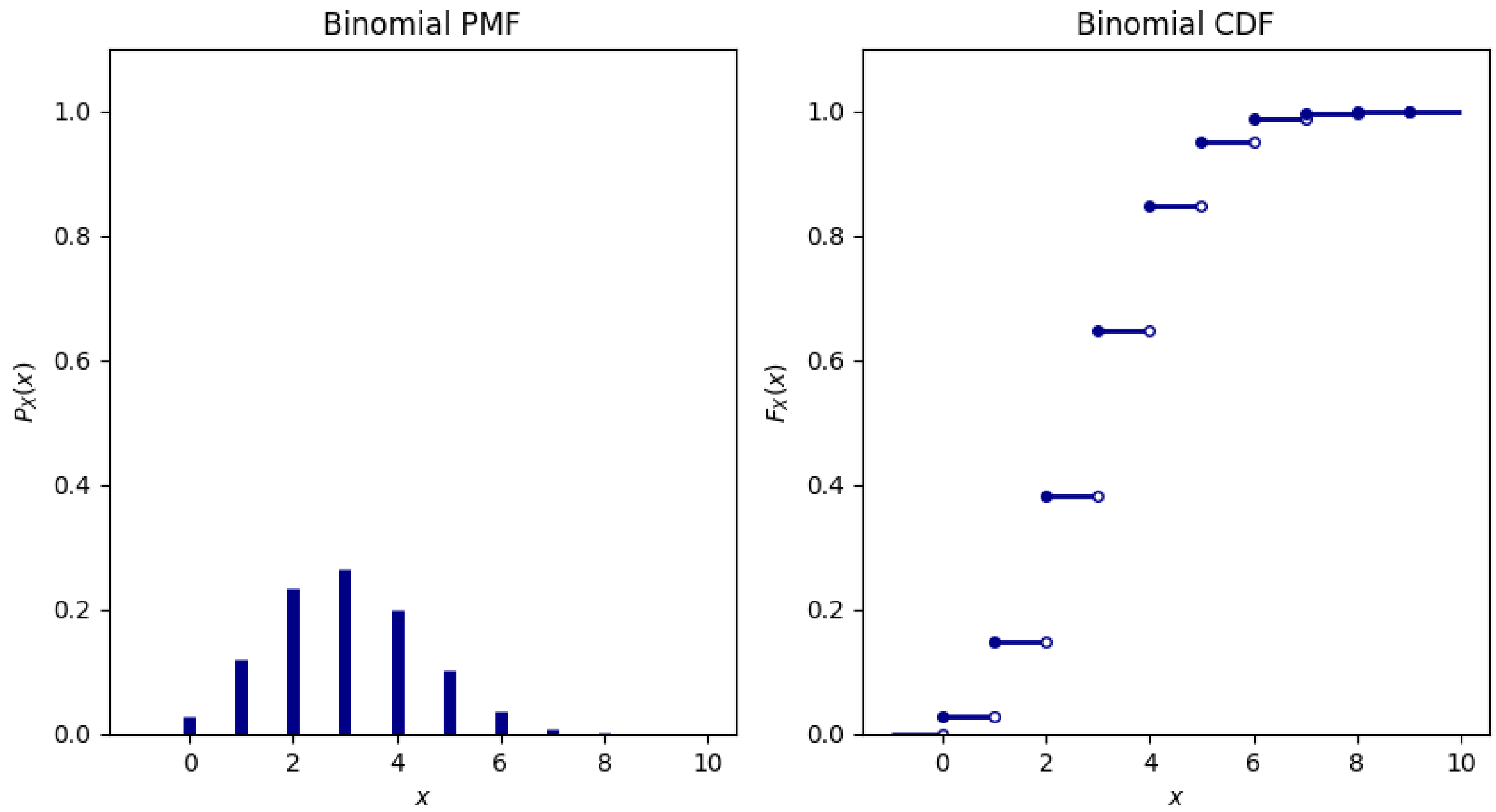
Binomial Distribution

The **Binomial** CDF is defined as

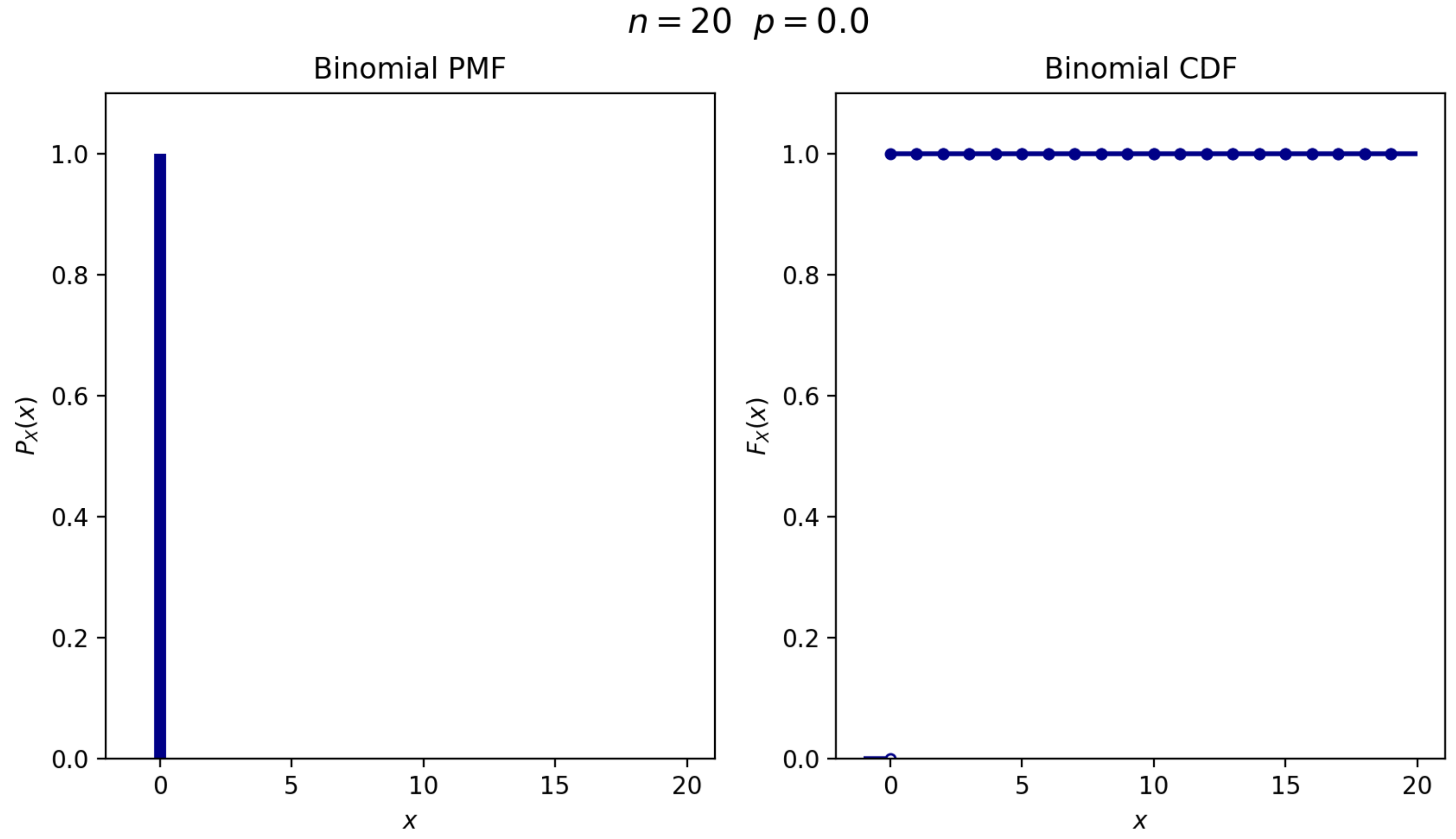
$$
\begin{aligned}
F_X(x) &= P(X \le x) \\
&= \begin{cases} 0, & x < 0, \\ \displaystyle\sum_{k=0}^{\lfloor x \rfloor} \binom{n}{k} p^k (1-p)^{n-k}, & 0 \le x < n, \\ 1, & x \ge n. \end{cases}
\end{aligned}
$$

Binomial Distribution

For $n = 10$ and $p = 0.3$



Binomial Distribution



Binomial Distribution

Expected Value

$$\mathrm{E}[X] = np$$

Variance

$$\mathrm{Var}[X] = np(1 - p)$$

Bernoulli Trials

- **Bernoulli Distribution (p):** Conducting a single Bernoulli Trial
- **Binomial Distribution (n, p):** Probability of observing *exactly* x successes in n trials
- **Geometric Distribution (p):**
- **Pascal Distribution (k, p):**

Geometric Distribution

If we want to know the probability that a series of **Bernoulli trials** requires x repetitions to achieve its *first success*, then we call the resulting distribution a **Geometric distribution**.

X is a **geometric** (p) random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where the parameter p is in the range $0 < p < 1$.

Geometric Distribution

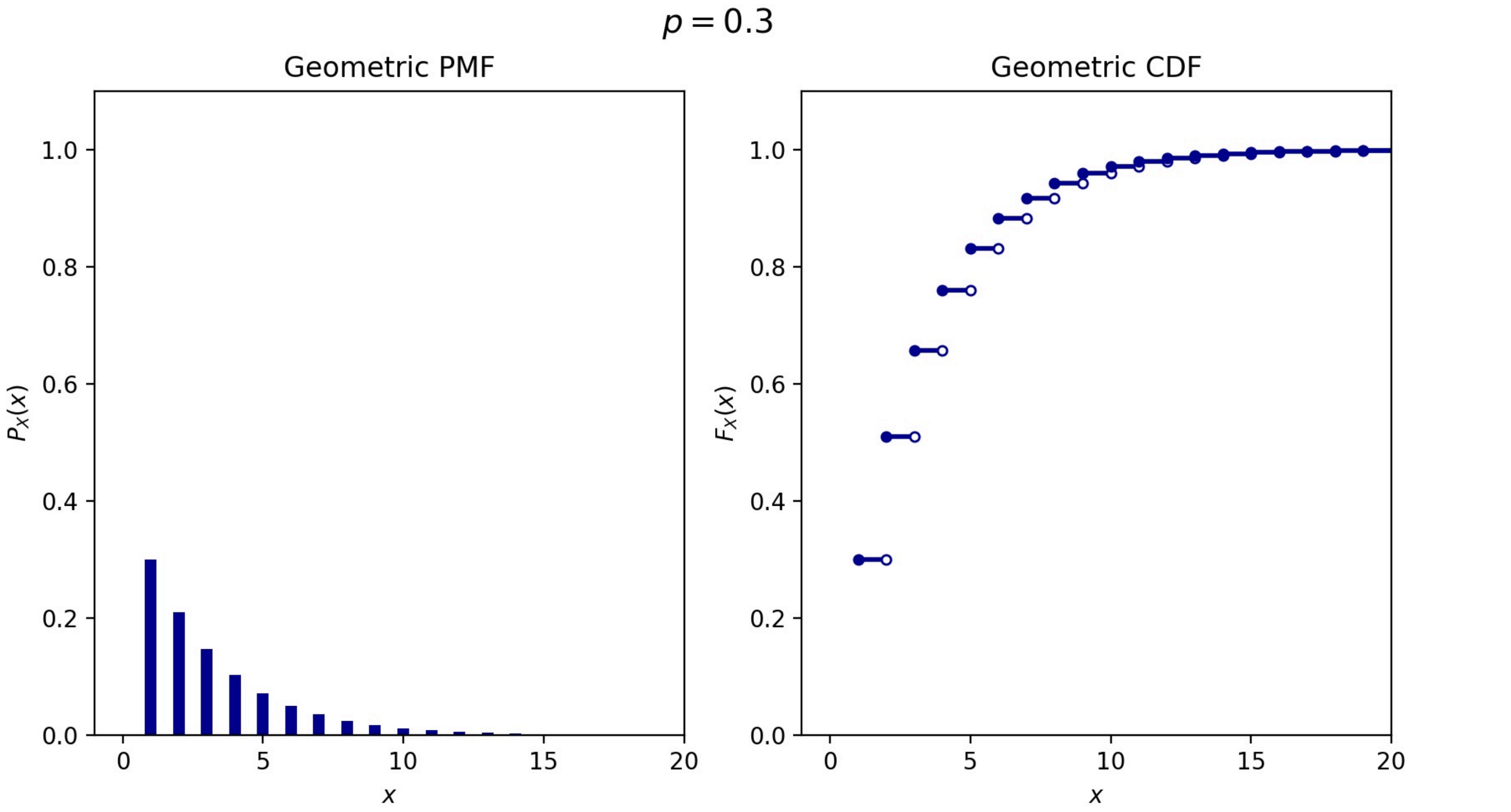
The **Geometric** CDF is defined as

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ 1 - (1-p)^{\lfloor x \rfloor} & x \geq 1 \end{cases}$$

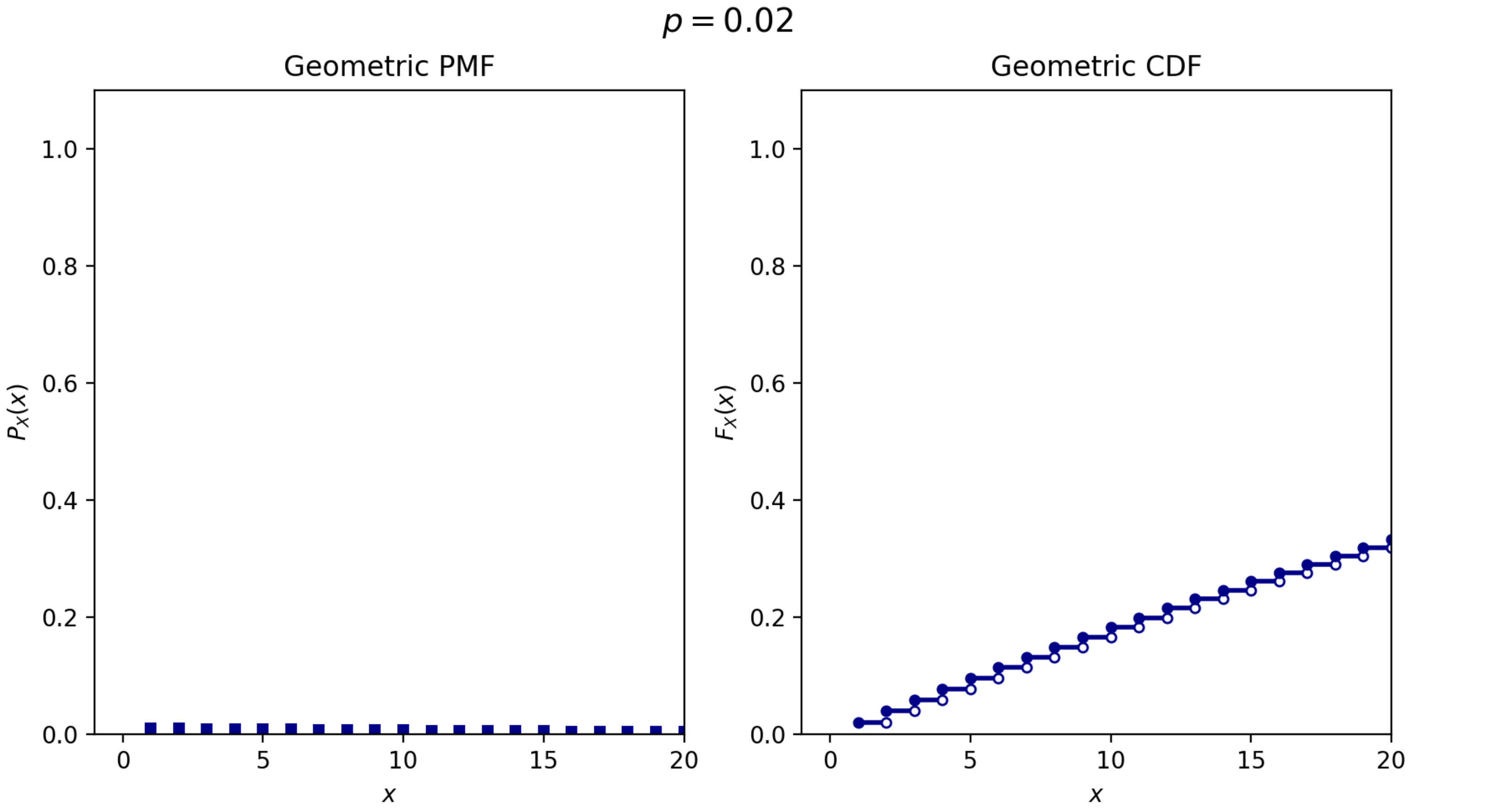
To calculate this from the PMF the following relation described by the *Finite Geometric Series* is used: $\sum_{j=1}^{n} x^{j-1} = \frac{1-x^n}{1-x}$

Geometric Distribution

For $p = 0.3$



Geometric Distribution



Geometric Distribution

Expected Value

$$\mathrm{E}[X] = \frac{1}{p}$$

Variance

$$\mathrm{Var}[X] = \frac{(1-p)}{p^2}$$

Variance increases for lower values of p.

Bernoulli Trials

- **Bernoulli Distribution (p):** Conducting a single Bernoulli Trial
- **Binomial Distribution (n, p):** Probability of observing *exactly* x successes in n trials
- **Geometric Distribution (p):** Probability of requiring x trials until seeing the *first* success
- **Pascal Distribution (k, p):** (next lecture)

Example 3.8

In a sequence of independent tests of integrated circuits, each circuit is rejected with probability p. Let Y equal the number of tests up to and including the first test that results in a reject. What is the PMF of Y?

Example 3.16

In Example 3.8, let the probability that a circuit is rejected equal $p = 1/4$. The PMF of Y, the number of tests up to and including the first reject, is the geometric $(1/4)$ random variable with PMF

$$P_Y(y) = \begin{cases} (1/4)(3/4)^{y-1} & y = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

What is the CDF of Y?