Systems and Signals 344

2026

Lecture 7

Variance and Standard Deviation

Section 3.8

Recap

Expected Value

The **expected value** of a **discrete** random variable X is

$$\mathrm{E}[X] = \mu_X = \sum_{x \in S_X} x P_X(x)$$

Recap

Functions of a Random Variable

Steps:

1. Perform experiment and observe an outcome s.
2. From s, find x, the corresponding value of random variable X.
3. Observe y by calculating $y = g(x)$.
4. For all possible values of $x \in S_X$ we compute $y = g(x)$ to get S_Y
.

For a discrete random variable X, the PMF of $Y = g(X)$ is

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

Expected Value of a Derived Random Variable

If $Y = g(X), \mathrm{E}[Y]$ *can be calculated from* $P_X(x)$ *and* $g(X)$ *without deriving* $P_Y(y)$.

Given a random variable X *with PMF* $P_X(x)$ *and the derived random variable* $Y = g(X)$, *the expected value of* Y *is*

$$\mathrm{E}[Y] = \mu_Y = \mathrm{E}[g(X)] = \sum_{x \in S_X} g(x) P_X(x)$$

(note that only the part in front of $P_X(x)$ *changed)*

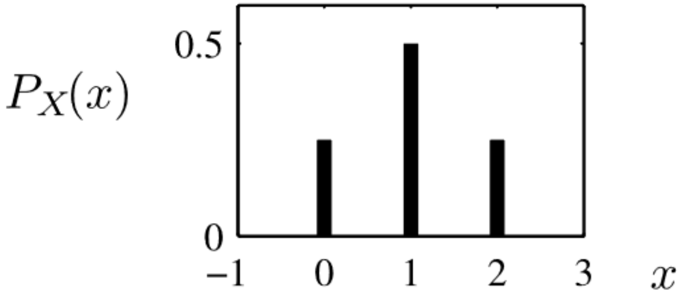
Properties of Expected Values

For any random variable X

- $\mathrm{E}[aX + b] = a\mathrm{E}[X] + b$
  - This is often called a **linear** transformation. But if $b \neq 0$ then it should technically be called **affine** since the origin is not fixed.
- $\mathrm{E}[X - \mu_X] = 0$
- Usually, $\mathrm{E}[g(X)] \neq g(\mathrm{E}[X])$

Example 3.23

Recall from Examples 3.5 and 3.18 that X has PMF



$$P_X(x) = \begin{cases} 1/4 & x = 0, \\ 1/2 & x = 1, \\ 1/4 & x = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.61}$$

What is the expected value of $V = g(X) = 4X + 7$?

From Theorem 3.12,

$$\mathrm{E}[V] = \mathrm{E}[g(X)] = \mathrm{E}[4X + 7] = 4\,\mathrm{E}[X] + 7 = 4(1) + 7 = 11. \tag{3.62}$$

We can verify this result by applying Theorem 3.10:

$$\begin{aligned} \mathrm{E}[V] &= g(0)P_X(0) + g(1)P_X(1) + g(2)P_X(2) \\ &= 7(1/4) + 11(1/2) + 15(1/4) = 11. \end{aligned} \tag{3.63}$$

Variance and Standard Deviation

The variance $\mathbf{Var}[X]$ *measures the dispersion (or spread) of sample values of* X *around the expected value* $\mathbf{E}[X]$.

Variance and Standard Deviation

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$$\mathbf{Var}[X] = \mathbf{E}[(X - \mu_X)^2]$$

It is important to note that $\mathbf{Var}[X]$ does not have the same units as X and can therefore not be drawn on the same graphical axes.

Variance and Standard Deviation

Variance

$$\mathrm{Var}[X] = \mathrm{E}[(X - \mu_X)^2]$$

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mu_X^2$$

Variance and Standard Deviation

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If X has for example units of m, then variance has units of m^2.
So we can't compare variance with the $\mathrm{E}[X]$...

Variance and Standard Deviation

Variance

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$$\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mu_X^2$$

$$\mathrm{Var}[X] = \sigma_X^2$$

Standard Deviation

$$\sigma_X = \sqrt{\mathrm{Var}[X]}$$

Variance and Standard Deviation

Standard Deviation

$$\sigma_X = \sqrt{\mathrm{Var}[X]}$$

- This is quite useful since σ_X has the same units as X.
- Therefore, the standard deviation can be compared with the expected value.
- For a Gaussian random variable, approximately 68.3% of the probability lies in $[\mu_X - \sigma_X, \mu_X + \sigma_X]$. No fixed proportion applies to every distribution.

Variance and Standard Deviation

- In many applications we try to make a prediction $\hat{x}$ before performing an experiment and observing a sample value of X (therefore often called the **blind estimate** of X).
- We want to minimise the squared prediction error, and one of the most common criteria used to determine $\hat{x}$ is the **mean squared error (MSE)**:

$$e = \mathrm{E}[(X - \hat{x})^2]$$

In the absence of observations, the minimum mean square error estimate of random variable X is:

$$\hat{x} = \mathrm{E}[X] = \mu_X$$

Variance and Standard Deviation

In the absence of observations, the minimum mean square error estimate of random variable X is:

$$\hat{x} = \mathrm{E}[X] = \mu_X$$

$$e = \mathrm{E}[(X - \hat{x})^2] = \mathrm{E}[(X - \mu_X)^2] = \mathrm{Var}[X]$$

Properties of Variance and Standard Deviation

- Since $(X - \mu_X)^2 \geq 0$ then $\mathbf{Var}[X] \geq 0$
- $\mathbf{Var}[aX + b] = a^2 \, \mathbf{Var}[X]$
  - Note that b falls away, since shifting a random variable by a constant does not change the dispersion of outcomes around the expected value.
  - For $a = 0$ then $\mathbf{Var}[b] = 0$ since there is no dispersion around the expected value of a constant.
- Since $\mathbf{Var}[aX] = a^2 \, \mathbf{Var}[X]$ then $\sigma_{aX} = |a|\sigma_X$

Moments

The **moments** of X summarise aspects of its distribution, but they do not always fully describes the distribution. However, the *moment generating function* does fully describe the distribution.

Moments are quantitative measures related to the shape of a function's graph.

The n^{th} **moment about the origin** of X is, $g_n(x) = x^n$

$$m_n = \mathrm{E}[X^n] = \sum_{x \in S_X} x^n P_X(x)$$

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The n^{th} **central moment** of X is, $g_n(x) = (x - \mu_X)^n$

$$\mu_n = \mathrm{E}[(X - \mu_X)^n] = \sum_{x \in S_X} (x - \mu_X)^n P_X(x)$$

(not to be confused with the expected value μ_X)

This is also sometimes referred to as the **moment about the mean**

Moments

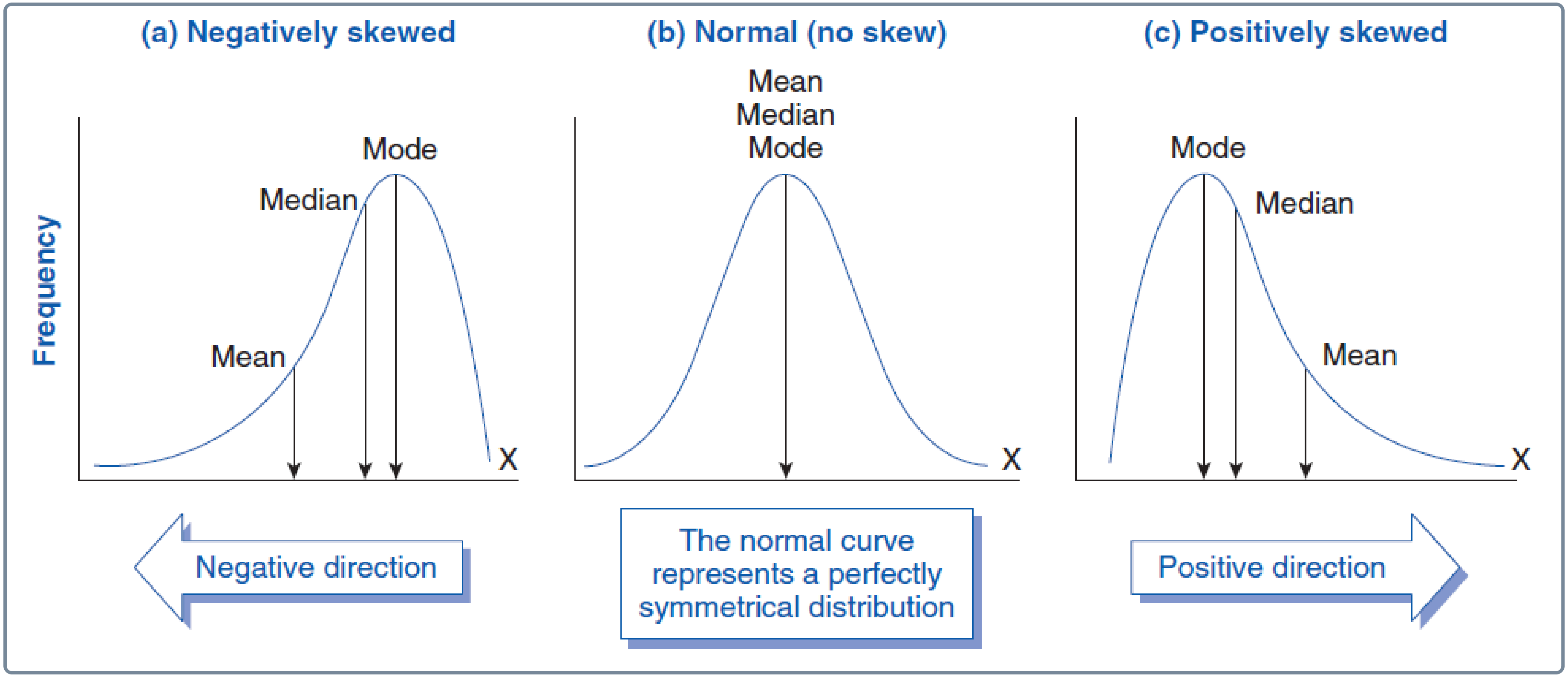
Moment about the origin

1. $m_1 = \mathrm{E}[X] = \mu_X$
2. $m_2 = \mathrm{E}[X^2]$

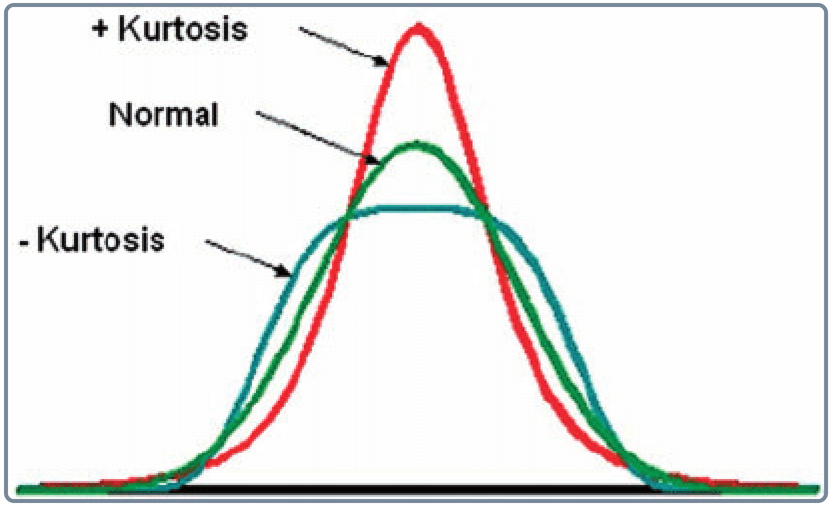
Central moments

0. $\mu_0 = \mathrm{E}[(X - \mu_X)^0] = 1$
1. $\mu_1 = \mathrm{E}[(X - \mu_X)^1] = 0$
2. $\mu_2 = \mathrm{E}[(X - \mu_X)^2] = \mathrm{Var}[X] = \sigma_X^2$
3. $\mu_3 = \mathrm{E}[(X - \mu_X)^3]$ is called **Skew**
   - Indicates *asymmetry* about the mean
   - Often expressed as **cofficient of skewness** $\gamma = \mu_3/\sigma^3$
4. $\mu_4 = \mathrm{E}[(X - \mu_X)^4]$ is called **Kurtosis**
   - Is a measure of *tailedness*
   - Usually normalised $\kappa = \mu_4/\sigma^4$

$$\gamma = \frac{\mathrm{E}[(X - \mu_X)^3]}{\sigma_X^3}$$ is the **coefficient of skewness**

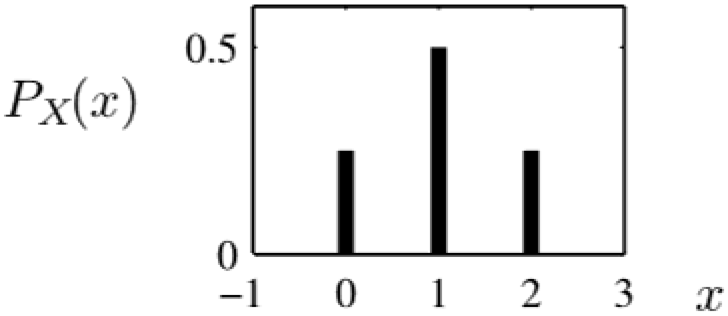


$$\kappa = \frac{\mathrm{E}[(X-\mu_X)^4]}{\sigma_X^4}$$ is the **kurtosis**



Example 3.25

Continuing Examples 3.5, 3.18, and 3.23, we recall that X has PMF

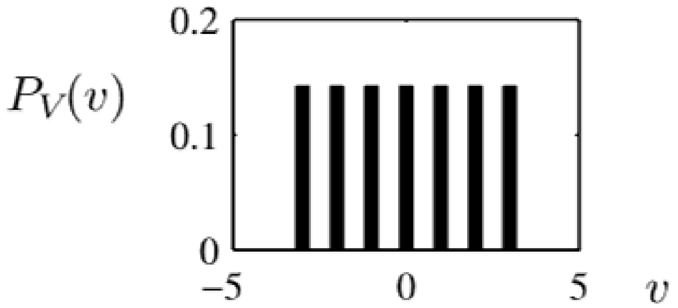


$$P_X(x) = \begin{cases} 1/4 & x = 0, \\ 1/2 & x = 1, \\ 1/4 & x = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and expected value $\mathrm{E}[X] = 1$. What is the variance of X?

Example 3.27

The amplitude V (volts) of a sinusoidal signal is a random variable with PMF



$$P_V(v) = \begin{cases} 1/7 & v = -3, -2, \ldots, 3, \\ 0 & \text{otherwise.} \end{cases}$$

A new voltmeter records the amplitude U in millivolts. Find the variance and standard deviation of U.