Systems and Signals 344

2026

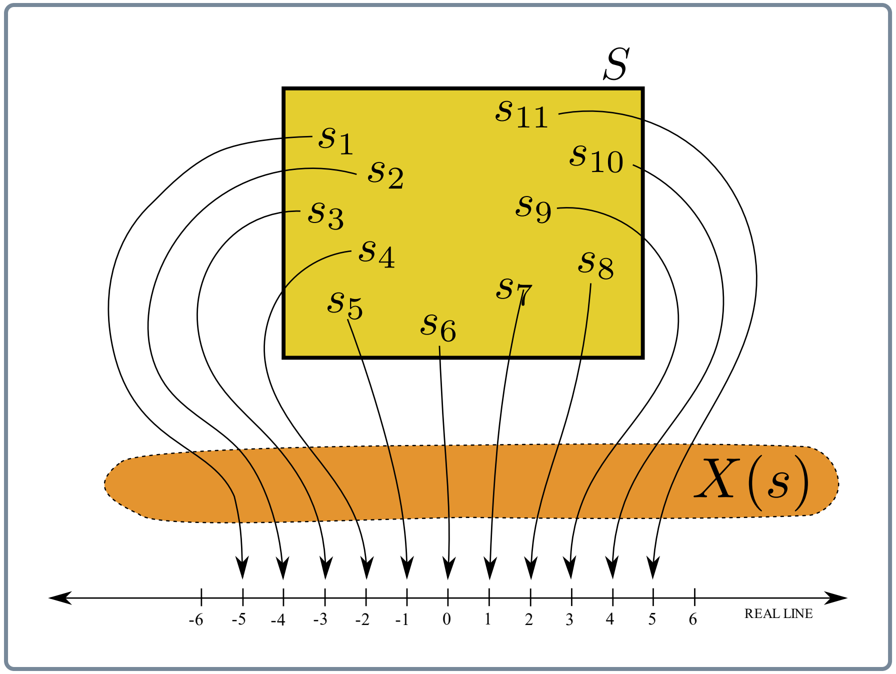
Lecture 6

Cumulative Distribution Functions (CDF);
Averages and Expected Values;
Functions of a Random Variable

Sections 3.4-3.7

Recap - Random Variables

A ***random variable*** $X(s)$ *is a function mapping each outcome* s *(of a sample space* S*) to a real-numbered value. Each of these values has an associated probability of being observed.*



Recap - Probability Mass Function (PMF)

The Probability Mass Function (PMF) of ***discrete*** *random variable X expresses the probability model of an experiment as a mathematical function. The function is the probability $P(X = x)$ for every number x.*

$$P_X(x) = P(X = x)$$

In simple terms: The PMF indicates the probability that a discrete random variable X will be *exactly equal* to a specific value x.

Cumulative Distribution Function (CDF)

Like the PMF, the CDF of a ***discrete*** *or* ***continuous*** *random variable X expresses the probability model of an experiment as a mathematical function. The function is the probability $P(X \leq x)$ for every number x.*

Cumulative Distribution Function (CDF)

Like the PMF, the CDF of any ***discrete*** *or* ***continuous*** *random variable X expresses the probability model of an experiment as a mathematical function. The function is the probability $P(X \leq x)$ for every number x.*

$$F_X(x) = P(X \leq x)$$

($F_X(x)$ is read as "the CDF of X evaluated at x")

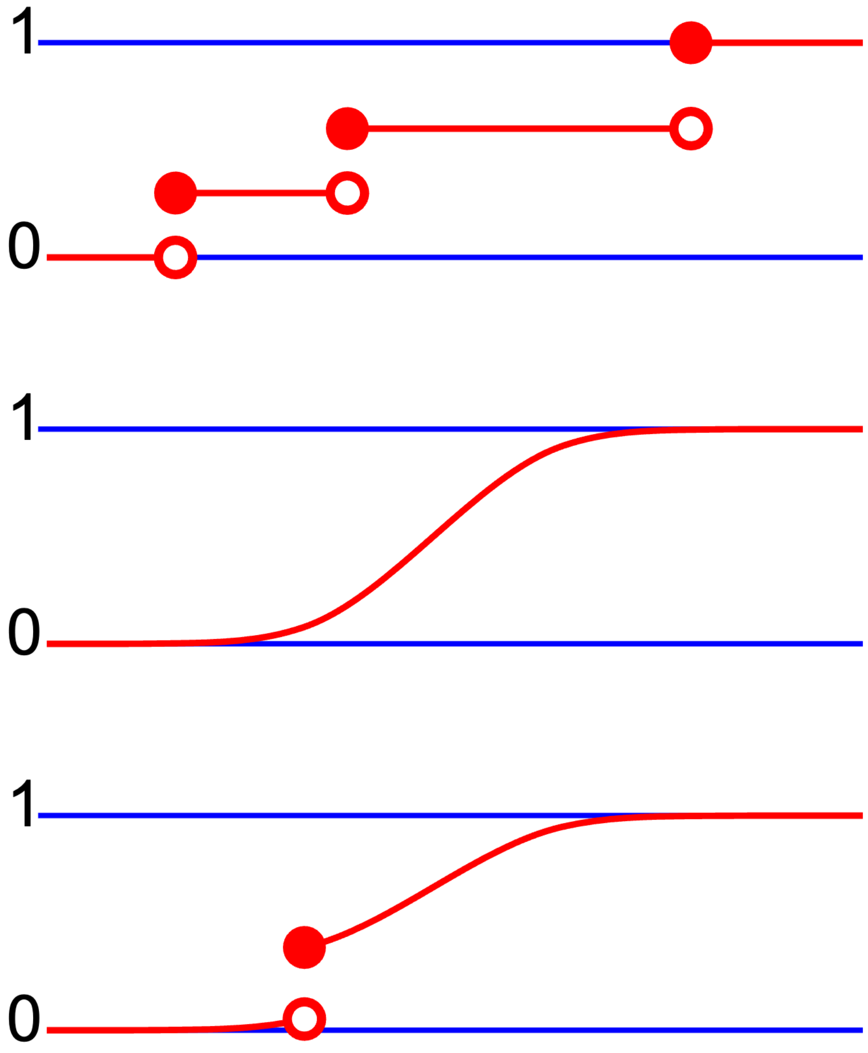
In simple terms: The CDF indicates the probability that a random variable X will be *less than or equal* to a specific value x.

Properties of CDFs

For any discrete random variable X *with range* $S_X = \{x_1, x_2, \ldots\}$*,*
satisfying $x_1 \le x_2 \le \ldots$*:*

1. $F_X(-\infty) = 0$ *and* $F_X(\infty) = 1$ *and* $0 \le F_X(x) \le 1$
2. *For all* $x_i \le x_j$, $F_X(x_i) \le F_X(x_j)$
   ($F_X(x)$ *is non-decreasing)*
3. *For* $x_i \in S_X$*, the jump at* x_i *is* $F_X(x_i) - F_X(x_i^-) = P_X(x_i)$
4. $F_X(x) = F_X(x_i)$ *for all* x *such that* $x_i \le x < x_{i+1}$

Cumulative Distribution Function (CDF)



CDF of Discrete Random Variable

CDF of Continuous Random Variable

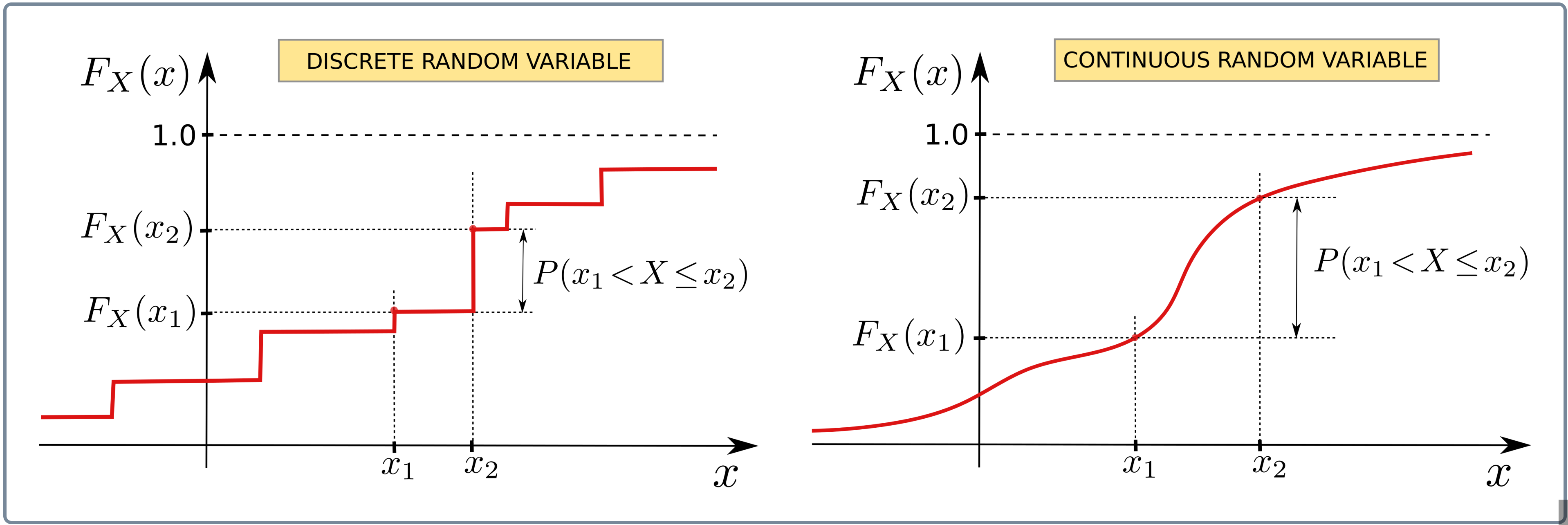
CDF of Random Variable with both discrete and continuous parts

Cumulative Distribution Function (CDF)

Properties of CDFs

For all $x_j \geq x_i$ *then*

$$P(x_i < X \leq x_j) = F_X(x_j) - F_X(x_i)$$



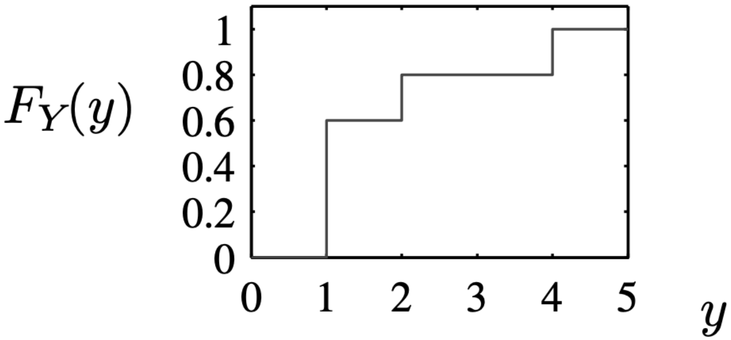
Proof

For $\boldsymbol{b} \geq \boldsymbol{a}$, prove that:

$$F_X(b) - F_X(a) = P(a < X \leq b)$$

Quiz 3.4

Use the CDF $F_Y(y)$ to find the following probabilities:



(a) $P[Y < 1]$

(b) $P[Y \leq 1]$

(c) $P[Y > 2]$

(d) $P[Y \geq 2]$

(e) $P[Y = 1]$

(f) $P[Y = 3]$

Averages and Expected Values

An average is a number that describes a set of experimental observations.

The expected value is a number that describes the probability model of an experiment.

Note on notation: $\mathbf{E}[X] = \mathbb{E}\{X\}$ (I will use the former)

Averages

- The **average** is a single number that is taken as representative of a set of n numbers.
    - It is called a *statistic* of the set of numbers.
        - (a fact or piece of data obtained from a study of a large quantity of numerical data.)

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  - It is called a *statistic* of the set of numbers.
    - (a fact or piece of data obtained from a study of a large quantity of numerical data.)
- There are several kinds of averages. The most common ones are the **mean**, the **median**, and the **mode**.
  - The **mean** value of n numbers is the sum of the n numbers divided by n.
  - The **median** is the number in the middle of a data set.
  - The **mode** is the most common number in the set.

Averages and Expected Values

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    - We assume that the probability models are related to the numbers observed in practice.

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- An **Average** is a *statistic* that describes a *set of numbers* observed in practice (describing a *sample* of the population).
- An **Expected Value** is a *parameter* that describes a *probability model* (describing the *entire* population).

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- The **probability model** of a random variable characterises an experiment with numerical outcomes.
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- An **Average** is a *statistic* that describes a *set of numbers* observed in practice (describing a *sample* of the population).
- An **Expected Value** is a *parameter* that describes a *probability model* (describing the *entire* population).
    - The expected value is a number that can be computed from the PMF or CDF of a random variable.
    - If the probability model is related to an application that results in a set of numbers, then the expected value of the random variable corresponds to the mean value of the set of numbers.

Expected Value

Provided the sum converges absolutely, the **expected value** of a **discrete** random variable X is

$$\mathrm{E}[X] = \mu_X = \sum_{x \in S_X} x P_X(x)$$

$\Rightarrow$ Sum of outcomes multiplied by the probability of each outcome.

- **Expectation** is a synonym for expected value.
- μ_X is alternative notation for the expected value of random variable X.
- The sample mean is commonly written as $\bar{X}$ (which does *not* mean the complement of X).

Examples

- Prove that the mean is the same as the expected value.
- Calculate the expected value for the basketball example.

Functions of a Random Variable

Often we want to use the observed sample values of a random variable to compute other quantities.

A function $Y = g(X)$ of random variable X is another random variable. The PMF $P_Y(y)$ can be derived from $P_X(x)$ and $g(X)$.

Functions of a Random Variable

Each sample value y of a ***derived random variable*** *Y is a mathematical function $g(x)$ of a sample value x of another random variable X. We adopt the notation $Y = g(X)$ to describe the relationship of the two random variables.*

$P_X(x)$ is associated with X, and we can use $Y = g(X)$ to get $P_Y(y)$.

Example 3.19

A parcel shipping company offers a charging plan: \$1.00 for the first pound, \$0.90 for the second pound, etc., down to \$0.60 for the fifth pound, with rounding up for a fraction of a pound. For all packages between 6 and 10 pounds, the shipper will charge \$5.00 per package. (It will not accept shipments over 10 pounds.) Find a function $Y = g(X)$ for the charge in cents for sending one package.

Functions of a Random Variable

Steps:

1. Perform experiment and observe an outcome s.
2. From s, find x, the corresponding value of random variable X.
3. Observe y by calculating $y = g(x)$.
4. For all possible values of $x \in S_X$ we compute $y = g(x)$ to get S_Y .

For a discrete random variable X, the PMF of $Y = g(X)$ is

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

(sum all the probabilities of x where $g(x) = y$)

Functions of a Random Variable

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If $g(X)$ transforms different values of x into different values of y (i.e. $g(x_i) \neq g(x_j)$ if $x_i \neq x_j$) then:

$$P_Y(y) = P[Y = g(x)] = P[X = x] = P_X(x)$$

Expected Value of a Derived Random Variable

If $Y = g(X), \mathrm{E}[Y]$ *can be calculated from* $P_X(x)$ *and* $g(X)$ *without deriving* $P_Y(y)$.

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Given a random variable X *with PMF* $P_X(x)$ *and the derived random variable* $Y = g(X)$*, the expected value of* Y *is*

$$\mathrm{E}[Y] = \mu_Y = \mathrm{E}[g(X)] = \sum_{x \in S_X} g(x) P_X(x)$$

(note that only the part in front of $P_X(x)$ *changed)*

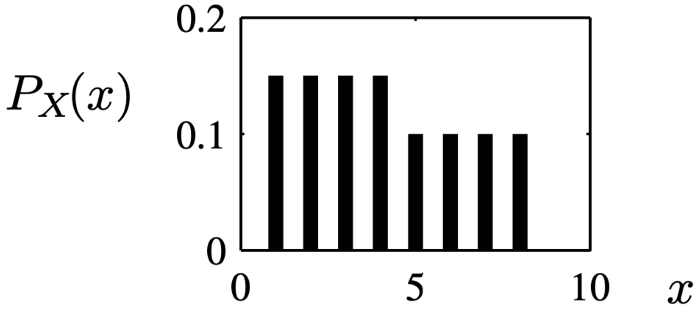
Example 3.20

In Example 3.19, suppose all packages weigh 1, 2, 3, or 4 pounds with equal probability. Find the PMF and expected value of Y, the shipping charge for a package.

Given that $\boldsymbol{Y = 105X - 5X^2}$ from the previous question

Example 3.21

Suppose the probability model for the weight in pounds X of a package in Example 3.19 is



$$P_X(x) = \begin{cases} 0.15 & x = 1,2,3,4, \\ 0.1 & x = 5,6,7,8, \\ 0 & \text{otherwise.} \end{cases}$$

For the pricing plan given in Example 3.19, what is the PMF and expected value of Y, the cost of shipping a package?