Systems and Signals 344

2026

Lecture 5

Discrete Random Variables and Probability Mass Functions

Sections 3.1-3.2

"Becoming" an Engineer

This might cause some emotions:

You need to LEARN...

- Uncomfortable
  - How to rise above it. Being "ok" with being uncomfortable.
- Unsure
  - How to take risks - just try something
- Impostor syndrome - not good enough
  - Don't compare. You have unique value
- Misunderstood
  - How to communicate
- Frustrated
  - How to adapt and persevere

Random Variables

- Previously we looked at a *physical* model of an experiment (procedure and observation).
- We also started with the *mathematical* probability model.
  - S is the sample space
  - Rule to assign numbers between 0 and 1 to a set $A \subseteq S$.
- However, not all outcomes are clearly numbers, but it is more convenient to perform statistical analysis using numbers (such as calculating averages).
  - $\longrightarrow$ Random Variables 😄
- A **random variable** assigns numbers to outcomes in the sample space of an experiment.

Random Variables

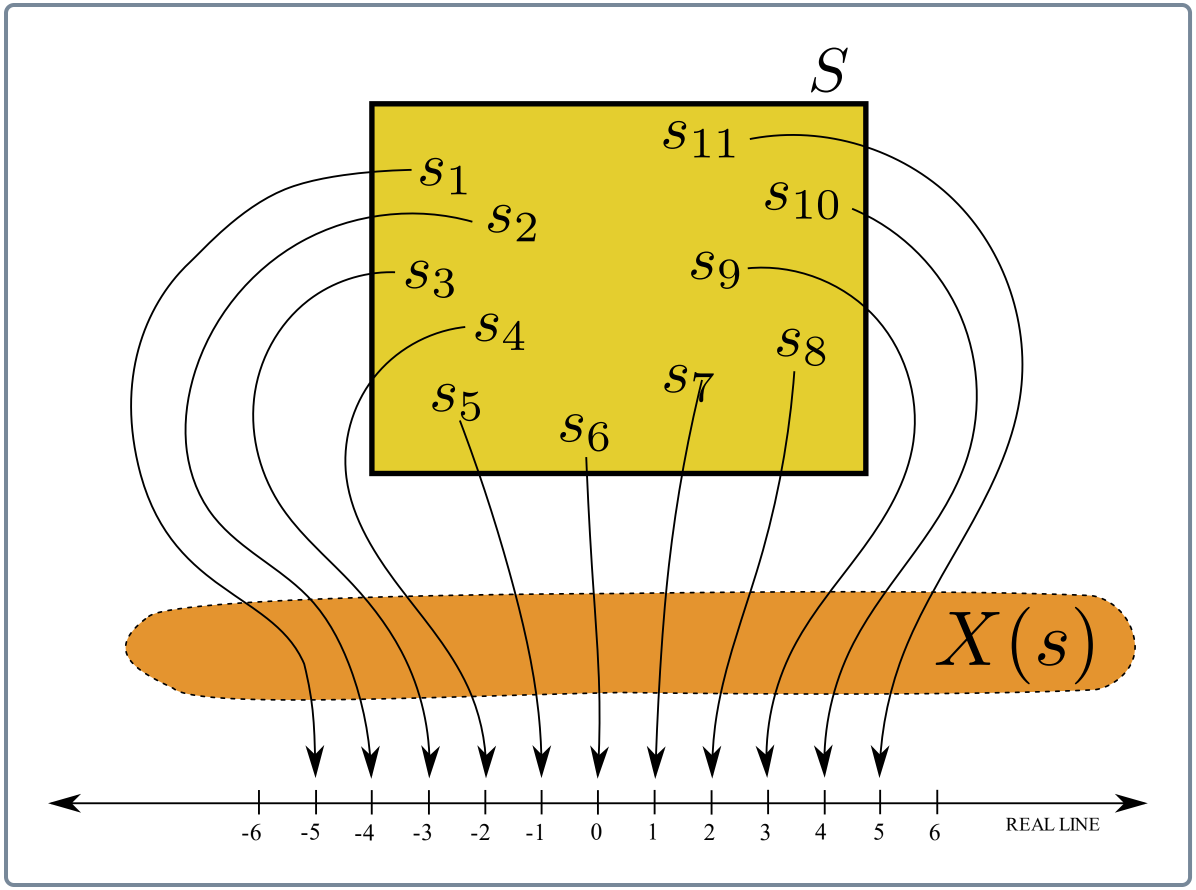
*A **random variable** $X(s)$ is a function mapping each outcome $\boldsymbol{s}$ (of a sample space S) to a real-numbered value. Each of these values has an associated probability of being observed.*

Random Variables

Misconceptions

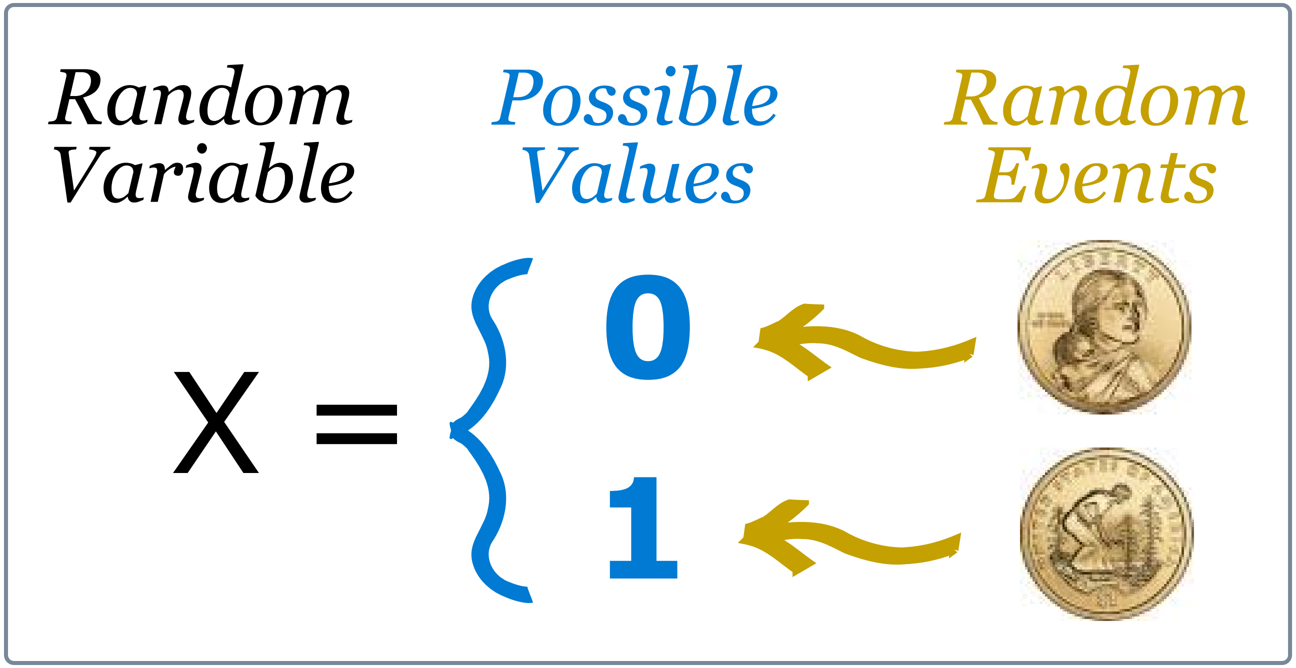
- A random variable is not like an algebraic variable which can be solved for (e.g. find x), nor like a programming variable that stores a specific value.
    - A random variable is a *function* that defines a mapping from the outcomes of a random experiment to a *set of values*.

Random Variables



Important: every outcome must map to a *single* finite real number, but more than one outcome may map to the *same* real number.

Random Variables



$$S_X = \{0, 1\}$$

Note: We could choose Heads=100 and Tails=150 or other values if we want! It is our choice.

Random Variables - Notation

- A random variable is represented by a capital letter (e.g. X or Y). A particular *value* of the random variable is represented by the corresponding lowercase letter (e.g. x or y).

- Random variables are functions that map outcome s to a value x. In other words, $X(s) = x$. We often simplify this to $X = x$.

- We can also write an *event* as, for example $\{X \leq x\}$ to emphasise that there is a set of outcomes $s \in S$ that maps to real numbers less than or equal to x (i.e. $X(s) \leq x$). This can also be written as: $\{X \leq x\} = \{s \in S \mid X(s) \leq x\}$

- The set of possible values that random variable X can take is called the **range** (or **support**) S_X.
  Dependent on how you define X.

Random Variables

There are three ways to relate a random variable to an experiment.

1. The random variable is the observation.
    - X is the number of a single die roll $S_X = \{1, 2, 3, 4, 5, 6\}$
2. The random variable is a function of the observation.
    - X is the number of heads in 3 coin tosses, e.g. $S_X = \{0, 1, 2, 3\}$
3. The random variable is a function of another random variable.
    - For example, $X = 10 - 2Y$

Random Variables

There are three ways to relate a random variable to an experiment.

1. The random variable is the observation.
    - X is the number of a single die roll $S_X = \{1, 2, 3, 4, 5, 6\}$
2. The random variable is a function of the observation.
    - X is the number of heads in 3 coin tosses, e.g. $S_X = \{0, 1, 2, 3\}$
3. The random variable is a function of another random variable.
    - For example, $X = 10 - 2Y$

- It is common, but not required, to derive the value of a random variable directly from the outcome of the underlying experiment.
- Experiments with parameter settings that do not produce random variables in *some* situations are called **improper experiments**. (See Example 3.4)

Example 3.4

The procedure of an experiment is to fire a rocket in a vertical direction from Earth's surface with initial velocity V km/h. The observation is T seconds, the time elapsed until the rocket returns to Earth. Under what conditions is the experiment improper?

. . . .

At low velocities, V, the rocket will return to Earth at a random time T seconds that depends on atmospheric conditions and small details of the rocket's shape and weight. However, when $V > v^* \approx 40{,}000$ km/hr, the rocket will not return to Earth. Thus, the experiment is improper when $V > v^*$ because it is impossible to perform the specified observation.

Random Variables

- The random variable X may be **discrete** or **continuous**.
- A **discrete random variable**
  - can take on a **finite** or a **countably infinite** number of values, and
  - these values are represented by a set of discrete points along the real axis.
  - $\Rightarrow$ X is a **discrete** random variable if the range of X is a countable set $S_X = \{x_1, x_2, \ldots\}$
- A **continuous random variable**
  - always takes on an **uncountably infinite** number of values, and
  - these values are represented by a continuum of points along the real axis.

Discrete Random Variables

Probability Mass Function (PMF)

*The Probability Mass Function (PMF) of a **discrete** random variable X expresses the probability model of an experiment as a mathematical function. The function is the probability $P(X = x)$ for every number x.*

Probability Mass Function (PMF)

The Probability Mass Function (PMF) of a ***discrete*** *random variable* X *expresses the probability model of an experiment as a mathematical function. The function is the probability* $P(X = x)$ *for every number* x*.*

$$P_X(x) = P(X = x)$$

In simple terms: The PMF indicates the probability that a discrete random variable X will be *exactly equal* to a specific value x.

Probability Mass Function (PMF)

Example 3.5

When the basketball player Wilt Chamberlain shot two free throws, each shot was equally likely either to be good (g) or bad (b). Each shot that was good was worth 1 point. What is the PMF of X, the number of points that he scored?

Probability Mass Function (PMF)

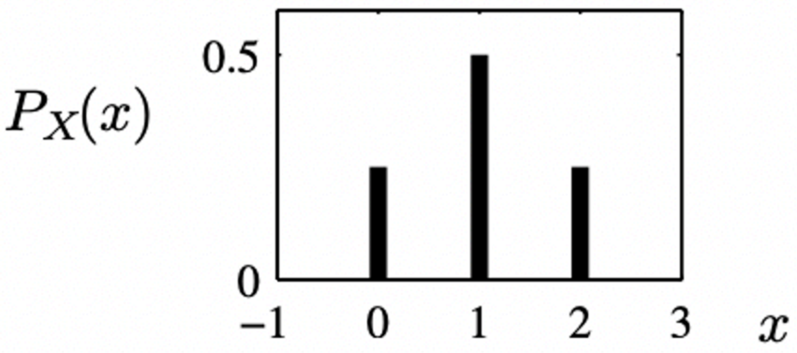
Definition of X

Outcomes	bb	bg	gb	gg
$\mathrm{P}[\cdot]$	1/4	1/4	1/4	1/4
X	0	1	1	2

Definition of P_X

$$P_X(x) = \begin{cases} 1/4 & x = 0, \\ 1/2 & x = 1, \\ 1/4 & x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Graph and table representation of P_X



x	0	1	2
$P_X(x)$	1/4	1/2	1/4

Properties of PMF

Axioms applied to PMF

For a discrete random variable X with PMF $P_X(x)$ and range S_X:

1. *For any x,* $P_X(x) \geq 0$
2. $\sum_{x \in S_X} P_X(x) = 1$
3. *For any event $B \subseteq S_X$, the probability that X is in the set B is*

$$P(B) = \sum_{x \in B} P_X(x)$$

Quiz 3.2

The random variable N has PMF

$$P_N(n) = \begin{cases} c/n & n = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Find

(a) The value of the constant c

(b) $\mathrm{P}[N = 1]$

(c) $\mathrm{P}[N \geq 2]$

(d) $\mathrm{P}[N > 3]$

Cumulative Distribution Function (CDF)

Like the PMF, the CDF of a ***discrete*** *or* ***continuous*** *random variable X expresses the probability model of an experiment as a mathematical function. The function is the probability $P(X \leq x)$ for every number x.*

Cumulative Distribution Function (CDF)

Like the PMF, the CDF of a ***discrete*** *or* ***continuous*** *random variable* X *expresses the probability model of an experiment as a mathematical function. The function is the probability* $P(X \leq x)$ *for every number* x*.*

$$F_X(x) = P(X \leq x)$$

($F_X(x)$ *is read as "the CDF of* X *evaluated at* x*")*

In simple terms: The CDF indicates the probability that a random variable X will be *less than or equal* to a specific value x.

Cumulative Distribution Function (CDF)