Systems and Signals 344

2026

Lecture 4

Sequential Experiments

Chapter 2

Tree Diagrams

- Many experiments consist of a sequence of **subexperiments**.
- The procedure followed for each subexperiment may depend on the results of the previous subexperiments.
- It is often useful to represent this with a type of graph called a **tree diagram.**

Important:

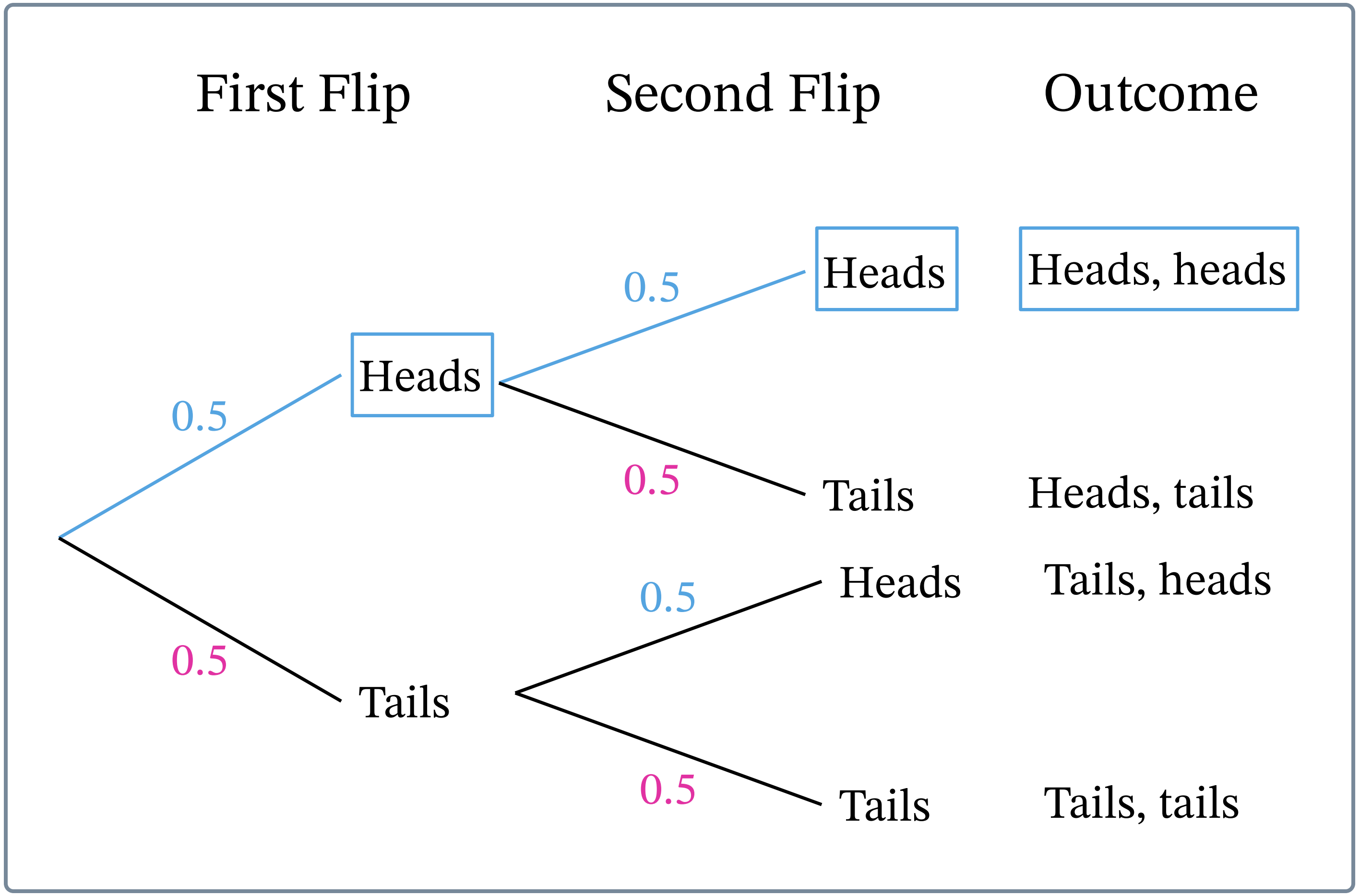
- It assumes that the sample space S is partitioned at the root of the tree.
- The probabilities of the branches leaving *any* node add up to 1.

Tree Diagrams

Tree diagrams *display the outcomes of the subexperiments in a sequential experiment. The labels of the branches are probabilities and conditional probabilities.*

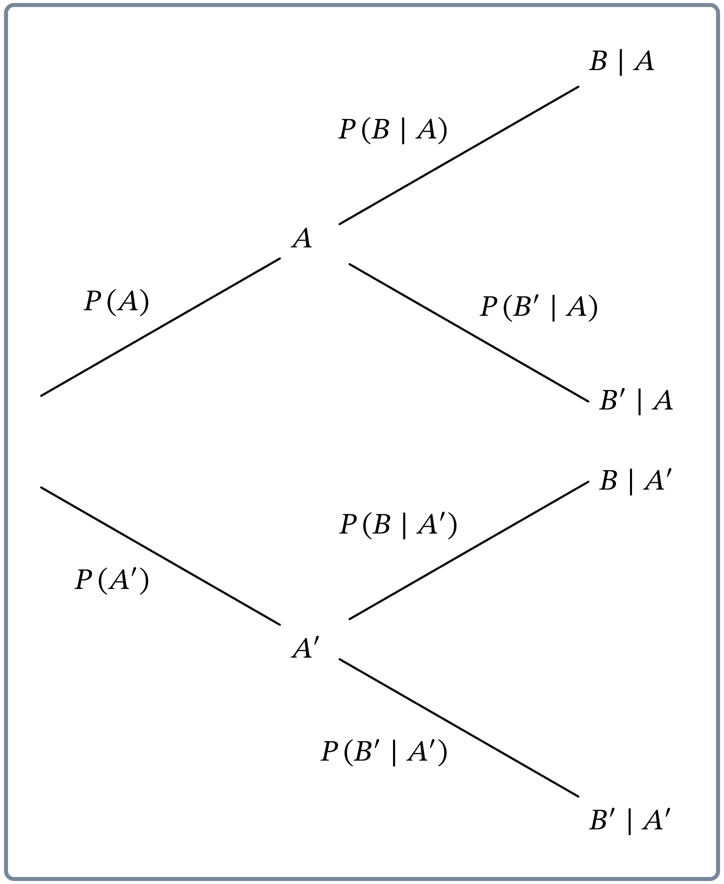
The probability of an outcome of the entire experiment is the product of the probabilities of branches going from the root of the tree to a leaf.

Tree Diagrams



Tree Diagrams

The labels of the branches of the second subexperiment are the **conditional probabilities** of the events in that subexperiment.



Rule of Thumb

You can only multiply probabilities if:

- the events are **independent**:
  $P(A \cap B) = P(A) \cdot P(B)$
- *or* the **second is conditioned on the first** (events are dependent):
  $P(A \cap B) = P(A) \cdot P(B \mid A)$

Example 2.1

Previously:

Example 1.16

A company has three machines B_1, B_2, and B_3 making 1 kΩ resistors. Resistors within 50 Ω of the nominal value are considered acceptable. It has been observed that 80% of the resistors produced by B_1 and 90% of the resistors produced by B_2 are acceptable. The percentage for machine B_3 is 60%. Each hour, machine B_1 produces 3000 resistors, B_2 produces 4000 resistors, and B_3 produces 3000 resistors. All of the resistors are mixed together at random in one bin and packed for shipment. What is the probability that the company ships an acceptable resistor?

Results:

$$P(B_1) = \frac{3}{10},\ P(B_2) = \frac{4}{10},\ P(B_3) = \frac{3}{10}$$

$$P(A|B_1) = 0.8,\ P(A|B_2) = 0.9,\ P(A|B_3) = 0.6$$

Now:

What is the probability of drawing a resistor from B_2 that is not acceptable?

Example

A box contains $\mathbf{6}$ red balls and $\mathbf{9}$ green balls for a total of $\mathbf{15}$ balls. You draw $\mathbf{3}$ balls *without replacement*.

Calculate the following:

- $P(\textbf{third ball red} \mid \textbf{first two red})$
- $P(\textbf{third ball red} \mid \textbf{first two different})$
- $P(RGR)$
- $P(RGR \cup GRR)$

Counting Methods

In all applications of probability theory it is important to understand the sample space of an experiment. Counting methods can be used to determine the number of outcomes in the sample space of a sequential experiment.

Counting Methods

- Consider selecting k items randomly from a set of n **distinguishable objects**.
  - This is called **sampling**, and the k chosen items form a **sample**.

Counting Methods

- Consider selecting k items randomly from a set of n **distinguishable objects**.
    - This is called **sampling**, and the k chosen items form a **sample**.
- We can do this *with/without* replacement and *noting/not noting* order.
- Select **without replacement** and order **is** important (*sequence*)
  $\Rightarrow$ **Permutation**
- Select **without replacement** and order **is not** important (*set*)
  $\Rightarrow$ **Combination**

Fundamental Principle of Counting

An experiment consists of two subexperiments. If one subexperiment has n outcomes and the other subexperiment has m outcomes, then the experiment has $n \cdot m$ outcomes.

Permutations

The **permutation** P_k^n is the number of different **sequences** of k elements that can be drawn from a set of n different elements *without replacement*.

$$P_k^n = n \cdot (n-1) \cdot (n-2) \dots (n-k+1) = \frac{n!}{(n-k)!}$$

Note that sometimes different notation is used. For example:

$$P_k^n = (n)_k$$

Example: Permutations

Draw 4 cards from a 52-card deck. Put them *in the order drawn*.
How many sequences exist?

Combination

- The **combination** $\binom{n}{k}$ (read as "n choose k") is the number of different **sets** of k elements that can be drawn from a set of n elements *without replacement*.
- To obtain the combination, the permutation must be reduced by the number of ways we can order k items.
  - Or, realising that permutation can be thought of as a first subexperiment with $\binom{n}{k}$ possible outcomes, and a second experiment with $P_k^k = k!$ possible outcomes. Combined there are P_k^n possible outcomes.
  - $\Rightarrow P_k^n = \binom{n}{k} \cdot k!$
- Thus:

$$\binom{n}{k} = \frac{P_k^n}{P_k^k} = \frac{n!}{(n-k)!\,k!}$$

Example: Combinations

I have four objects, A, B, C, D. I select two objects *without replacement* and put the two letters in *alphabetical order*.

- How many unique outcomes do I have?
- How would the result differ if the order did matter?

Combination

- Note that choosing k out of n elements to be part of a subset is equivalent to choosing $(n-k)$ elements to be excluded from the subset:

$$\binom{n}{k} = \binom{n}{n-k}$$

- For a nonnegative integer n and any integer k, we use the convention $\binom{n}{k} = 0$ when $k < 0$ or $k > n$.

Combination

The numbers $\binom{n}{k}$ are also known as the **binomial coefficients**, since they are used in the **Binomial Expansion Formula**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Self-study: Look at how this relates to Pascal's Triangle.

Example 2.9

There are four queens in a deck of 52 cards. You are given seven cards at random from the deck. What is the probability that you have no queens?

Sampling with Replacement

- Sampling *with replacement* refers to the process where the selected object is placed back into the set before making another selection.
- These subexperiments are referred to as **independent trials**.
  - It effectively corresponds to performing k repetitions of an identical subexperiment.
  - The probability models of all the subexperiments are *identical and independent* of the outcomes of previous subexperiments.

> *Given n distinguishable objects, there are n^k ways to choose with replacement an ordered sample of k objects.*

Previous example:

Example 2.9

There are four queens in a deck of 52 cards. You are given seven cards at random from the deck. What is the probability that you have no queens?

Now:

How would the result differ if you draw seven cards *with replacement*?

Bernoulli Trials

- For an experiment with 2 possible outcomes, A and $\bar{A}$, with $P(A) = p$ and $P(\bar{A}) = 1 - p$.
- The experiment is repeated n times, and the trials are *statistically independent*. Such repetitions are known as **Bernoulli trials**.
- We might ask: What is the probability that A is observed exactly k times in n trials?

$$P(A_k) = \binom{n}{k} p^k (1-p)^{n-k}$$

- We will encounter this again when we discuss the **binomial distribution**.

Example 2.17

In Example 1.16, we found that a randomly tested resistor was acceptable with probability $\mathrm{P}[A] = 0.78$. If we randomly test 100 resistors, what is the probability of T_i, the event that i resistors test acceptable?

Testing each resistor is an independent trial with a success occurring when a resistor is acceptable. Thus for $0 \leq i \leq 100$,

$$\mathrm{P}\left[T_i\right] = \binom{100}{i} (0.78)^i (1 - 0.78)^{100-i} \tag{2.18}$$

We note that our intuition says that since 78% of the resistors are acceptable, then in testing 100 resistors, the number acceptable should be near 78. However, $\mathrm{P}[T_{78}] \approx 0.096$, which is fairly small. This shows that although we might expect the number acceptable to be close to 78, that does not mean that the probability of exactly 78 acceptable is high.