The Poisson Process

Example 13.8

Let $X(t) = \overset{\text{RV}}{R}\,|\cos(2\pi \overset{\text{Constant}}{f} \overset{\text{function variable}}{t})|$ be a rectified cosine signal having random amplitude R which has the exponential PDF:

$$f_R(r) = \begin{cases} \frac{1}{10} e^{-\frac{r}{10}} & r \geq 0 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \lambda = \frac{1}{10}$$

What is the PDF $f_{X(t)}(x)$?

$X(t) \geq 0$ for all t $\qquad x < 0: P(X(t) \leq x) = 0$

1. Get CDF
2. $\frac{d}{dx} \rightarrow$ PDF

$$F_R(r) = 1 - e^{-\lambda r}$$

1. $x \geq 0$:

$$F_{X(t)}(x) = P(X(t) \leq x) = P(R|\cos(2\pi f t)| \leq x)$$

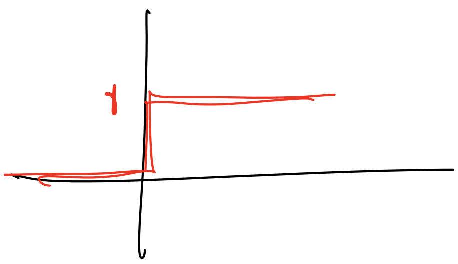
$$= P\left(R \leq \frac{x}{|\cos(2\pi f t)|}\right)$$

$$= \int_0^{\frac{x}{|\cos(2\pi f t)|}} \frac{1}{10} e^{-\frac{r}{10}} \, dr.$$

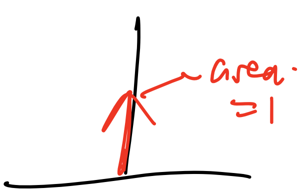
$\vdots$

$$= F_R\left(\frac{x}{|\cos(2\pi f t)|}\right)$$

CDF



PDF



$$F_{X(t)}(x) = \begin{cases} 1 - e^{-\frac{x}{10|\cos(2\pi f t)|}} & x \geq 0 \end{cases}$$

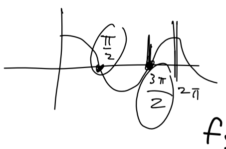
$$\begin{cases} 0 & x < 0 \end{cases}$$

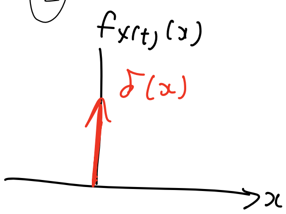
$\downarrow \frac{d}{dx}$

$$f_{X(t)}(x) = \frac{d F_{X(t)}(x)}{dx} = \begin{cases} \dfrac{1}{10|\cos(2\pi f t)|} e^{-\frac{x}{10|\cos(2\pi f t)|}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

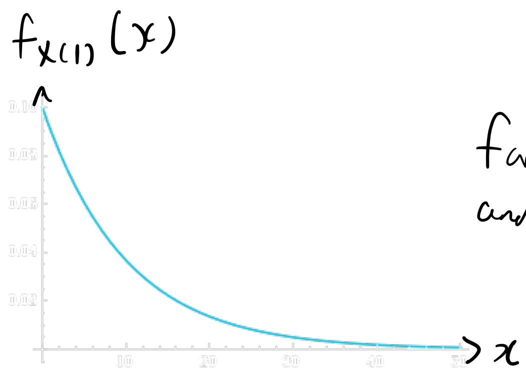
$\cos(2\pi f t) = 0$? $2\pi f t = \frac{\pi}{2} + k\pi$

$\frac{\pi}{2}(2n-1)$



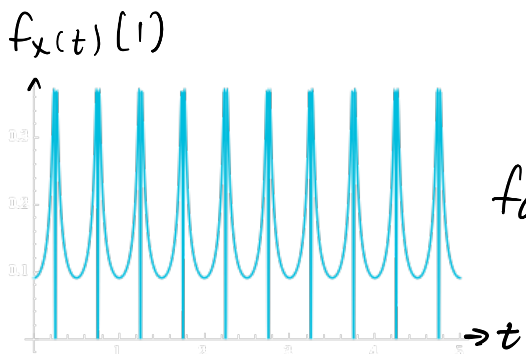


for $2\pi f t = \frac{\pi}{2} + \pi + 0{,}0$



for $t=1$
and $f=1$

↳ For fixed time, how likely to see each realization



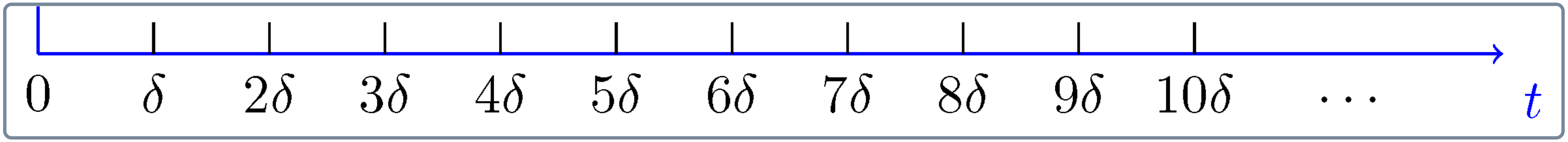
for $x = 1$ and $f = 1$

↳ one sample function

The Poisson Process

We can use a **Bernoulli process** $X_1, X_2, \ldots$ to derive a simple counting processing. We will then use this to describe the Poisson Process.

We now divide the half-line $[0, \infty)$ to tiny subintervals of length δ.



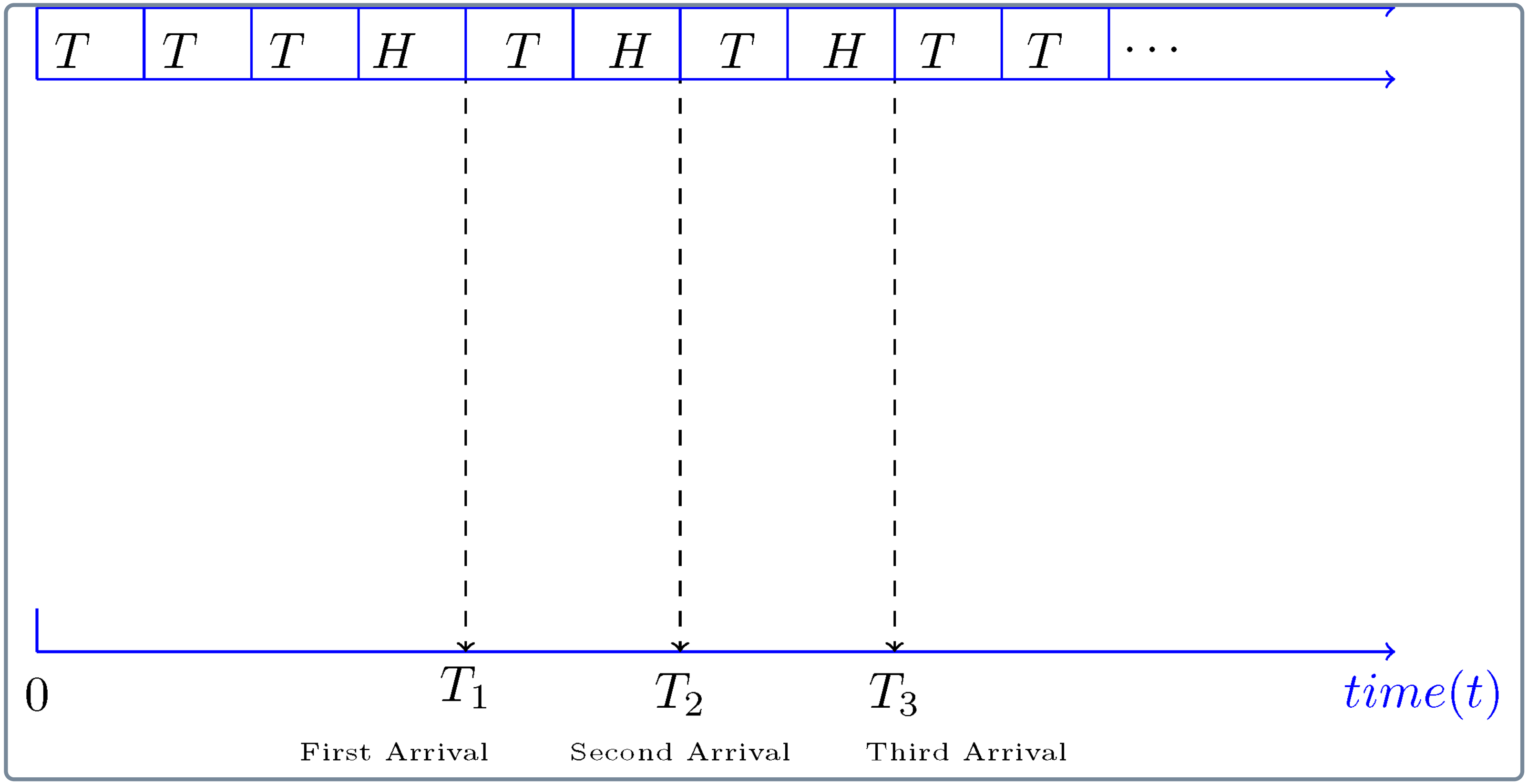
Each subinterval corresponds to a time slot of length δ. Thus, the intervals are $(0, \delta]$, $(\delta, 2\delta]$, $(2\delta, 3\delta]$, $\ldots$. More generally, the i-th interval is $((i-1)\delta, i\delta]$.

We assume that in each time slot, we, for example, toss a coin for which

$$P(X_k = 1) = P(\text{Success}) = P(\text{Heads}) = p = \lambda\delta.$$

The Poisson Process

If the coin lands heads up, we say that we have an arrival in that subinterval. Otherwise, we say that we have no arrival in that interval.



Here, we have an arrival at time $T = i\delta$, if the i-th coin flip results in a heads. (e.g. $T_1 = 4\delta, T_2 = 6\delta$)

The Poisson Process

Now, let N_m (a random sequence) be defined as the *number of arrivals* (number of heads) from time 0 to time T (i.e. $(0, T]$) with $T = m\delta$. Where m is the number of intervals (or number of coin flips).

e.g. $N_6 = 2$

\# intervals

We conclude that N_m is **Binomial**$(m, p = \lambda\delta = \frac{\lambda T}{m})$, with the following PMF:

$$P_{N_m}(n) = \binom{m}{n}\left(\frac{\lambda T}{m}\right)^n\left(1 - \frac{\lambda T}{m}\right)^{m-n}$$

Recall also that $\alpha = \lambda T$

Binomial$\left(m,\ p = \frac{\alpha}{m}\right)$ $\qquad \frac{m!}{n!\,(m-n)!}$

$$P_{N_m}(n) = \binom{m}{n}\left(\frac{\alpha}{m}\right)^{n}\left(1-\frac{\alpha}{m}\right)^{m-n}$$

$$= \frac{m(m-1)\cdots(m-n+1)}{m^{n}}\cdot\frac{\alpha^{n}}{n!}\left(1-\frac{\alpha}{m}\right)^{m-n}$$

$\frac{n \text{ terms}}{n \text{ terms}}$ $\quad$ each with form $\frac{m-j}{m}$

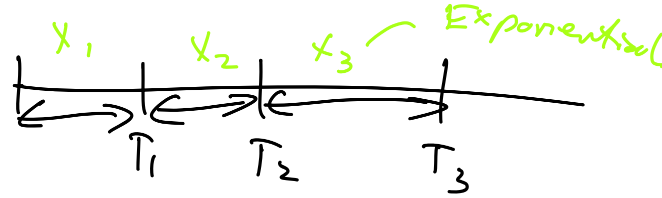
and $\lim_{m\to\infty}\frac{m-j}{m} = 1$

↳ or $\delta \to 0$

$$\therefore \lim_{m\to\infty}\frac{m(m-1)\cdots(m-n+1)}{m^{n}} = 1$$

$$\lim_{m\to\infty}\frac{\left(1-\frac{\alpha}{m}\right)^{m}}{\left(1-\frac{\alpha}{m}\right)^{n}} \qquad \begin{matrix}\to e^{-\alpha} \ \text{(identity)} \\ \to 1\end{matrix}$$

$$= e^{-\alpha}$$



$$\Rightarrow \lim_{m\to\infty} P_{N_m}(n) = \begin{cases}\dfrac{\alpha^{n}e^{-\alpha}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise}\end{cases}$$

Poisson RV $\qquad E[X] = \alpha = \lambda T$

$$= \lambda(t_1 - t_0)$$

$\boxed{\text{Note: } P(X_n > x) = 1 - P(X_n \leq 1) = 1 - F_{X_n}(x) = e^{-\lambda x}}$

$$P(X_n > t + x \mid X_n > t) = \frac{P(X_n > t + x \wedge X_n > t)}{P(X_n > t)}$$

$$= \frac{P(X_n > t + x)}{P(X_n > t)}$$

$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}}$$

$$= e^{-\lambda x} = P(X_n > x)$$

$$\lambda_1 = \lambda p \qquad \lambda_2 = \lambda(1-p)$$

$$\lambda = \lambda_1 + \lambda_2 = \lambda p + \lambda(1-p)$$

$$= \cancel{\lambda p} + \lambda - \cancel{\lambda p}$$

$$=$$

$$= \lambda$$

$$\lambda_1 = \lambda p = (\lambda_1 + \lambda_2) p$$

$$p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$