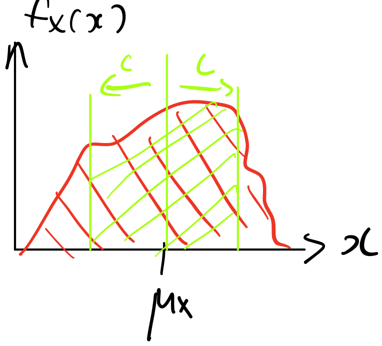
Estimates

$$P(|X-\mu_X| \geq c) \leq \frac{Var[X]}{c^2}$$



<u>Law of large numbers</u> (is $M_n(X) = E[X]$?) $n \to \infty$

Proof: $P(|M_n(X) - \mu_X| \geq c) \leq \dfrac{Var[X]}{nc^2}$

$$Y = M_n(X)$$

$$E[Y] = E[M_n(X)] = E[X] = \mu_X$$

$$Var[Y] = Var[M_n(X)] = \frac{Var[X]}{n}$$

$$P(|Y-\mu_Y| \geq c) \leq \frac{Var[Y]}{c^2} \qquad \text{Cheb.}$$

$$P(|M_n(X) - \mu_X| \geq c) \leq \underbrace{\frac{Var[X]}{nc^2}}_{n \to \infty} \to 0$$

$c \to \epsilon = 0{,}001$

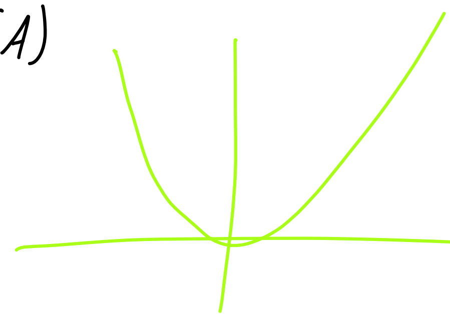
$$P(|\underbrace{M_n(X)}_{\hat{R}} - \mu_X| \geq \underbrace{0{,}001}_{\to 0}) \leq 0$$

r

$$\Rightarrow M_n(X) = \mu_X = E[X] \qquad n \to \infty$$

$$\boxed{E[\hat{R}] = \frac{n-1}{n}\, r}$$

$$P(A) = E[X] = \sum_{x \in S_X} x \cdot P_X(x) = 1 \cdot P(A)$$



Define

$$P_X(x) = \begin{cases} P(A) & x = 1 \quad \text{- A occurs} \\ 1 - P(A) & x = 0 \quad \text{- A did not occur.} \\ 0 & \text{otherwise} \end{cases}$$

From sampling:

estimate: $\hat{P}_n(A)$

true : $P(A)$

want $\hat{P}_n(A) = P(A)$ as $n \to \infty$

MSE proof

$$E[\hat{R}] = r = \mu_{\hat{R}} \leftarrow \text{unbiased}$$

$$\Rightarrow \underbrace{E[(\hat{R}-r)^2]}_{\text{MSE}} = E[\hat{R}^2 - 2\hat{R}r + r^2]$$

$$= E[\hat{R}^2] - 2rE[\hat{R}] + r^2$$

$$= E[\hat{R}^2] - 2r^2 + r^2$$

$$= E[\hat{R}^2] - r^2$$

$$= E[\hat{R}^2] - E[\hat{R}]^2$$

$$e_n = Var[\hat{R}]$$

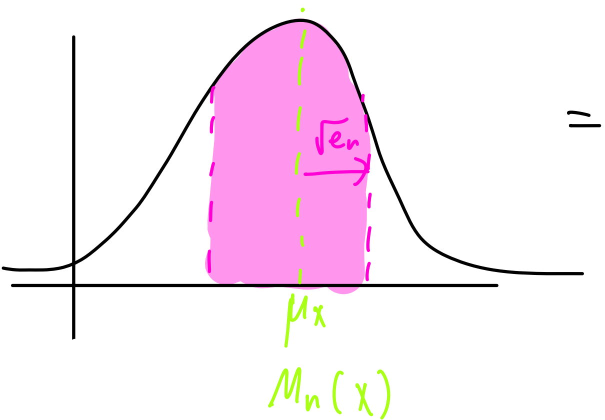
$$\underbrace{\sqrt{e_n}}_{\text{Standard error.}} = \sigma_{\hat{R}} = \sqrt{Var[\hat{R}]}$$

Standard error

If X is Gaussian: $P(\underbrace{E[X]}_{\mu_X} - \underbrace{\sqrt{e_n}}_{\sigma_X} \le M_n(X) \le E[X] + \sqrt{e_n})$

and $M_n(X)$ also Gaussian

$$\vdots$$



$$= 2\Phi(1) - 1 = 0{,}68 \longrightarrow$$

Example 10.6

Define

$$X_A = \begin{cases} 1 & \text{if } A \\ 0 & \text{otherwise} \end{cases}$$

$$P_{X_A}(x_A) = \begin{cases} P(A) & x_A = 1 \\ 1 - P(A) & x_A = 0 \\ 0 & \text{otherwise} \end{cases}$$

NOT probability
but a RV producing
2 possible values.

$$E[X] = \sum_{x \in S_X} x \cdot P_X(x)$$

Question: #n so that $\sqrt{e_n} \leq 0.1$? $e_n \leq 0.01$

$$e_n = E[(M_n(X_A) - E[X_A])^2] = Var[M_n(X_A)] = \frac{Var[X_A]}{n}$$

$$e_n = \frac{Var[X_A]}{n}$$

MSE

$$= \frac{p(1-p)}{n} \leq 0.01$$

$$Var[X_A] = E[X_A^2] - E[X_A]^2$$

$$= (1^2 \cdot P(A)) - (1 \cdot P(A))^2$$

$$= P(A)(1 - P(A))$$

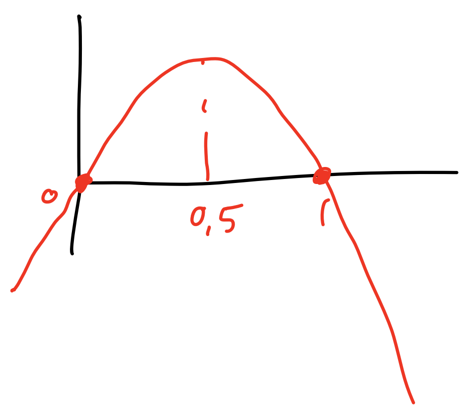
$$= p(1-p)$$

$$100\, p(1-p) \leq n$$

$$100\,(0.25) \leq n$$

$$25 \leq n$$

$n = 25$



$$\overset{R}{M_n(X)} = \overset{r}{E[X]}$$

$$V_n(X) \neq Var[X]$$

We want to estimate $r = Var[X]$

$$Var[M_n(X)] = \frac{Var[X]}{n}$$

$$M_n(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$Var[X] = E[(X-\mu_X)^2]$$

$$= \sum_{x \in S_X} (x-\mu_X)^2 \cdot P_X(x)$$

$$W = (X-\mu_X)^2$$

$$M_n(W) = \frac{1}{n}\sum_{i=1}^{n} W_i$$

$$= \frac{1}{n}\sum_{i=1}^{n}(X-\mu_X)^2 \rightarrow \text{Correctly estimates } Var[X]$$

$$\frac{1}{n-1}$$

If $Var[W]$ exists then $M_n(W)$ is a consistent and unbiased estimate

$$E[M_n(W)] = E[W] = \underbrace{E[(X-\mu_X)^2]}_{= Var[X].}$$

$$E[V_n(X)] = \frac{n-1}{n-1} Var[X]$$