Moment Generating Functions

$$\mathbb{C}_{XY} = \begin{bmatrix} Var[X] & Cov[X,Y] \\ Cov[Y,X] & Var[Y] \end{bmatrix}$$

$$Var[W_n] = \sum_{i=1}^{n} Var[X_i] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Cov[X_i, X_j]$$

$Cov[X_1, X_2]$, $Cov[X_1, X_3]$, $Cov[X_2, X_3]$

If $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$, the covariance matrix of $\mathbf{X}$ is

$$\mathbf{C_X} = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] & \mathrm{Cov}[X_1, X_3] \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Var}[X_2] & \mathrm{Cov}[X_2, X_3] \\ \mathrm{Cov}[X_3, X_1] & \mathrm{Cov}[X_3, X_2] & \mathrm{Var}[X_3] \end{bmatrix} \tag{8.23}$$

Example 9.2

$$X_i = \begin{cases} 1 & \text{if person } i \text{ draws own hat} \\ 0 & \text{otherwise} \end{cases}$$

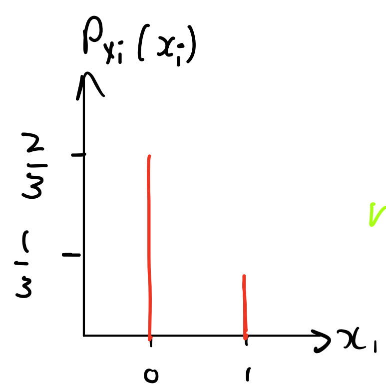
$$\boxed{S_{x_n} = [0, n]}$$

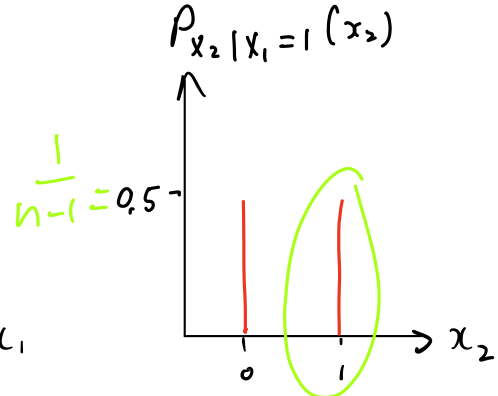
Number of matches $V_n = X_1 + X_2 + \dots X_n$

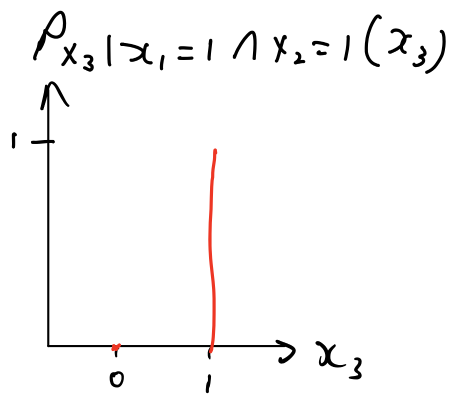
$$E[V_n] = E[X_1 + X_2 + \cdots + X_n]$$
$$= E[X_1] + E[X_2] + \cdots + E[X_n] = n \cdot \frac{1}{n} = 1$$

$$E[X_i] = \sum_{x \in S_{X_i}} x \, p_{X_i}(x) = \cancel{0 \cdot p_{X_i}(0)} + 1 \cdot p_{X_i}(1) = 1 \cdot \frac{1}{n} = \frac{1}{n} = E[X_i^2]$$

What is PMF of X_i (for $n=3$)?







PMF of person 1 drawing own hat — person 2 drawing own hat — person 3 drawing own hat

$$p_{X_i}(1) = \frac{1}{n} \qquad p_{X_i}(0) = 1 - \frac{1}{n} = \frac{n-1}{n}$$

$$Var[V_n] = \sum_{i=1}^{n} Var[X_i] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Cov[X_i, X_j]$$

$$= n\,Var[X_i] + n(n-1)\,Cov[X_i, X_j] \quad \text{iid.}$$

$$= 1 - \frac{1}{n} + 1 - \frac{n-1}{n}$$

$$= 2 + \frac{-1 \quad -n+1}{n} = 2 - 1 = 1 \longrightarrow$$

$$Var[X_i] = E[X_i^2] - E[X_i]^2$$

$$= \frac{1}{n} - \left(\frac{1}{n}\right)^2$$

$$Cov[X_i, X_j] = r_{x_i x_j} - \mu_{x_i}\mu_{x_j}$$

$$= E[X_i X_j] - E[X_i]E[X_j]$$

$$= \frac{1}{n(n-1)} - \frac{1}{n}\cdot\frac{1}{n} = \frac{1}{n(n-1)} - \frac{1}{n^2}$$

$$E[X_i X_j] = P_{X_i X_j}(x_i, x_j)\cdot 1\cdot 1 = P_{X_i|X_j}(1|1)\,P_{X_j}(1)$$

$$= \frac{1}{n-1}\cdot\frac{1}{n} = \frac{1}{n(n-1)}$$

$E[X]$

$E[X^2]$

$E[X^n]$

Moment Generating Functions

$$\phi_X(s) = E[e^{sX}]$$

$$E[X^n] = m_n = \frac{d^n \phi_X(s)}{ds^n}\bigg|_{s=0}$$

$$f(x) = \frac{\lambda}{x} \qquad g(s) = \lambda - s$$

$$f'(x) = -\frac{\lambda}{x^2} \qquad g'(s) = -1$$

Example 9.4

$$\phi_X(s) = \frac{\lambda}{\lambda - s}$$

$$\boxed{\text{Chain rule: } \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)}$$

$$E[X] = \frac{d\phi(s)}{ds}\bigg|_{s=0} = \frac{\lambda}{(\lambda - s)^2}\bigg|_{s=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$E[X^2] = \frac{d^2\phi(s)}{ds^2}\bigg|_{s=0} = \frac{2\lambda}{(\lambda - s)^3}\bigg|_{s=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$E[X^n] = \frac{d^n\phi(s)}{ds^n}\bigg|_{s=0} = \frac{n!\,\lambda}{(\lambda - s)^{n+1}}\bigg|_{s=0} = \frac{n!}{\lambda^n}$$

Example 9,5

Example 9.5

J and K are independent random variables with probability mass functions

$$\begin{array}{c|ccc} j & 1 & 2 & 3 \\ \hline P_J(j) & 0.2 & 0.6 & 0.2 \end{array}, \qquad \begin{array}{c|cc} k & -1 & 1 \\ \hline P_K(k) & 0.5 & 0.5 \end{array}. \tag{9.31}$$

Find the MGF of $M = J + K$. What are $P_M(m)$ and $E[M^3]$?

$$\phi_J(s) = E[e^{sJ}] = \sum_{j \in S_J} e^{sj} \cdot P_J(j)$$

$$= e^{s} \cdot 0{,}2 + e^{2s} \cdot 0{,}6 + e^{3s} \cdot 0{,}2$$

$$\phi_K(s) = E[e^{sk}] = e^{-s} \cdot 0{,}5 + e^{s} \cdot 0{,}5$$

$$\phi_M(s) = \phi_J(s)\phi_K(s) = 0{,}1 + 0{,}3e^{s} + 0{,}2e^{2s} + 0{,}3e^{3s} + 0{,}1e^{4s}$$

M	0	1	2	3	4
$P_M(m)$	0,1	0,3	0,2	0,3	0,1

$$E[M^3] = \left.\frac{d^3\phi_M(s)}{ds^3}\right|_{s=0} = \sum_{m \in S_M} m^3 \cdot P_M(m)$$

$$= 0{,}3e^{s} + 0{,}2 \cdot 2^3 e^{2s} + 0{,}3(3^3)e^{3s} + 0{,}1(4^3)e^{4s}\Big|_{s=0}$$

$$= \underline{16{,}4}$$