Random Vectors 2

A is a matrix $\quad$ $\bar{A}$ is a vector

$$E[\bar{X}] = \bar{\mu}_X = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$

$$\bar{X} = \underbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}_{2 \times 1} \qquad Y = \underbrace{\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}}_{3 \times 1}$$

$$\bar{X}\bar{Y}^T = \begin{array}{cc} \begin{bmatrix} X_1 Y_1 & X_1 Y_2 & X_1 Y_3 \\ X_2 Y_1 & X_2 Y_2 & X_2 Y_3 \end{bmatrix} & \begin{array}{l} X_1 \\ X_2 \end{array} \\ \begin{array}{ccc} Y_1 & \quad Y_2 & \quad Y_3 \end{array} & \end{array}$$

2×3

Definition (recap)

Correlation:

$$r_{X,Y} = E[XY]$$

Covariance:

$$Cov[X, Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

$$Cov[X, Y] = r_{X,Y} - \mu_X \mu_Y$$

Variance.

$$Cov[X, X] = Var[X] = E[(X-\mu_X)^2]$$

$$= \sigma_X^2$$

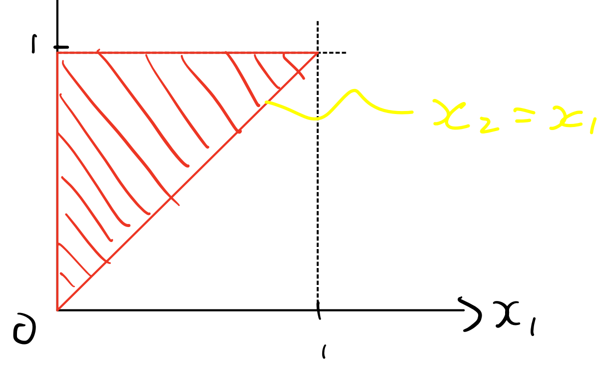
Correlation Coefficient

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} \qquad -1 \leq \rho_{XY} \leq 1$$

Note: $\mathbb{R}_{\bar{X}\bar{X}} = \mathbb{R}_{\bar{X}}$ and $\mathbb{C}_{\bar{X}\bar{X}} = \mathbb{C}_{\bar{X}}$

x_2

$\uparrow$



Example 8.7

Find the expected value $\mathrm{E}[\mathbf{X}]$, the correlation matrix $\mathbf{R}_{\mathbf{X}}$, and the covariance matrix $\mathbf{C}_{\mathbf{X}}$ of the two-dimensional random vector $\mathbf{X}$ with PDF

$$E[\bar{X}] = \begin{bmatrix} E[X_1] \\ E[X_2] \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 2 & 0 \le x_1 \le x_2 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.26}$$

The elements of the expected value vector are

$$\mathrm{E}\left[X_i\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_i f_{\mathbf{X}}(\mathbf{x})\, dx_1 dx_2 = \int_0^1 \int_0^{x_2} 2x_i \, dx_1 dx_2, \quad i = 1, 2. \tag{8.27}$$

The integrals are $\mathrm{E}[X_1] = 1/3$ and $\mathrm{E}[X_2] = 2/3$, so that $\boldsymbol{\mu}_{\mathbf{X}} = \mathrm{E}[\mathbf{X}] = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}'$. The elements of the correlation matrix are

$$\mathrm{E}\left[X_1^2\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^2 f_{\mathbf{X}}(\mathbf{x})\, dx_1 dx_2 = \int_0^1 \int_0^{x_2} 2x_1^2 \, dx_1 dx_2, \tag{8.28}$$

$$\mathrm{E}\left[X_2^2\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_2^2 f_{\mathbf{X}}(\mathbf{x})\, dx_1 dx_2 = \int_0^1 \int_0^{x_2} 2x_2^2 \, dx_1 dx_2, \tag{8.29}$$

$$\mathrm{E}\left[X_1 X_2\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{\mathbf{X}}(\mathbf{x})\, dx_1 dx_2 = \int_0^1 \int_0^{x_2} 2x_1 x_2 \, dx_1 dx_2. \tag{8.30}$$

These integrals are $\mathrm{E}[{X_1}^2] = 1/6$, $\mathrm{E}[{X_2}^2] = 1/2$, and $\mathrm{E}[X_1 X_2] = 1/4$.
Therefore,

$$\mathbf{R}_{\mathbf{X}} = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}. \tag{8.31}$$

We use Theorem 8.7 to find the elements of the covariance matrix.

$$\mathbf{C}_{\mathbf{X}} = \mathbf{R}_{\mathbf{X}} - \boldsymbol{\mu}_{\mathbf{X}} \boldsymbol{\mu}_{\mathbf{X}}' = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix} = \begin{bmatrix} 1/18 & 1/36 \\ 1/36 & 1/18 \end{bmatrix}. \tag{8.32}$$

Example 8.7

$$\mathbb{R}_{\bar{x}} = \begin{bmatrix} \mathrm{E}[X_1^2] & \mathrm{E}[X_1 X_2] \\ \mathrm{E}[X_2 X_1] & \mathrm{E}[X_2^2] \end{bmatrix}$$

$$\bar{\mu}_x \bar{\mu}_x^T = \underbrace{\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}}_{\substack{2\times1 \cdot 1\times2 \\ 2\times2}} = \begin{bmatrix} \frac{1}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{4}{9} \end{bmatrix}$$

$$\bar{Y} = A\bar{X} + \bar{b}$$

assume I have $\overline{\mu_x}$

Proof:

$$\bar{\mu}_Y = E[\mathbb{A}\bar{X} + \bar{b}] = \mathbb{A}E[\bar{X}] + \bar{b} = \mathbb{A}\bar{\mu}_X + \bar{b}$$

$$\bar{Y} - \bar{\mu}_Y$$
$$= \mathbb{A}\bar{X} + \bar{b} - (\mathbb{A}\bar{\mu}_X + \bar{b})$$
$$= \mathbb{A}(\bar{X} - \bar{\mu}_X)$$

$$\mathbb{C}_Y = \mathbb{A}\mathbb{C}_X\mathbb{A}^T$$
$$= E[(\bar{Y} - \bar{\mu}_Y)(\bar{Y} - \mu_Y)^T]$$
$$= E[(\mathbb{A}(\bar{X} - \bar{\mu}_X))(\mathbb{A}(\bar{X} - \mu_X))^T]$$
$$= E[\mathbb{A}(\bar{X} - \mu_X)(\bar{X} - \mu_X)^T\mathbb{A}^T]$$
$$= \mathbb{A}E[(\bar{X} - \bar{\mu}_X)(\bar{X} - \bar{\mu}_X)^T]\mathbb{A}^T$$
$$= \mathbb{A}\mathbb{C}_X\mathbb{A}^T$$

Example 8.9

$$\mathbb{R}_{\bar{X}} = \begin{bmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \qquad \mathbb{C}_{\bar{X}} = \begin{bmatrix} \frac{1}{18} & \frac{1}{36} \\ \frac{1}{36} & \frac{1}{18} \end{bmatrix} \qquad \mathbb{C}_{\bar{Y}} = \begin{bmatrix} \frac{1}{18} & \frac{5}{12} & \frac{1}{3} \\ \frac{5}{12} & 3{,}5 & 3{,}25 \\ \frac{1}{3} & 3{,}25 & 3{,}5 \end{bmatrix}$$

$$\bar{\mu}_X = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \qquad \mathbb{A} = \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$$

$$\bar{Y} = \mathbb{A}\bar{X} + \bar{b}$$

get $\mathbb{R}_{\bar{X}\bar{Y}}$, $\mathbb{C}_{\bar{X}\bar{Y}}$, ρ_{Y_1,Y_3}, ρ_{X_2,Y_1}

$$\mathbb{R}_{\bar{X}\bar{Y}} = \mathbb{R}_{\bar{X}}\mathbb{A}^T + \bar{\mu}_X \bar{b}^T$$

$$= \begin{bmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 & -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} & \frac{7}{4} & 2 \\ \frac{1}{4} & 3 & \frac{15}{4} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{2}{3} & -\frac{2}{3} \\ 0 & -\frac{4}{3} & -\frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} & \frac{13}{12} & \frac{4}{3} \\ \frac{1}{4} & \frac{5}{3} & \frac{29}{12} \end{bmatrix}$$

given:

$$\mathbb{C}_{\bar{X}} = \begin{bmatrix} \frac{1}{18} & \frac{1}{36} \\ \frac{1}{36} & \frac{1}{18} \end{bmatrix} \qquad \mathbb{A} = \begin{bmatrix} 1 & 0 \\ 6 & 3 \\ 3 & 6 \end{bmatrix}$$

$$\mathbb{C}_{\bar{X}\bar{Y}} = \mathbb{C}_{\bar{X}} \mathbb{A}^T$$

$$= \begin{bmatrix} \frac{1}{18} & \frac{1}{36} \\ \frac{1}{36} & \frac{1}{18} \end{bmatrix} \begin{bmatrix} 1 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{18} & \frac{5}{12} & \frac{1}{3} \\ \frac{1}{36} & \frac{1}{3} & \frac{5}{12} \end{bmatrix}$$

$$\rho_{Y_1, Y_3} = \frac{Cov[Y_1, Y_3]}{\sqrt{Var[Y_1]\, Var[Y_3]}} = \frac{\mathbb{C}_Y(1,3)}{\sqrt{\mathbb{C}_Y(1,1)\, \mathbb{C}_Y(3,3)}}$$

$$= \frac{\frac{1}{3}}{\sqrt{\frac{1}{18} \cdot 3{,}5}} = 0{,}76$$

$$\rho_{X_2, Y_1} = \frac{Cov[X_2, Y_1]}{\sqrt{Var[X_2]\, Var[Y_1]}} = \frac{\frac{1}{36}}{\sqrt{\frac{1}{18} \cdot \frac{1}{18}}}$$

$$\mathbb{C}_{XY} \quad \boxed{\mathbb{C}_X}\ \boxed{\mathbb{C}_Y}$$

$$= 0{,}5$$

Inverse of 2x2 :

1. Swap $\cdot$:
2. Negate $\cdot$
3. $\times \frac{1}{|A|}$

E.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $|A| = \underline{ad - bc}$

$$\Rightarrow A^{-1} = \frac{1}{|ad-bc|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 8.10

Consider the outdoor temperature at a certain weather station. On May 5, the temperature measurements in units of degrees Fahrenheit taken at 6 AM, 12 noon, and 6 PM are all Gaussian random variables, X_1, X_2, X_3, with variance 16 degrees^2. The expected values are 50 degrees, 62 degrees, and 58 degrees respectively. The covariance matrix of the three measurements is

X_1 X_2 X_3

$$\mathbf{C_X} = \begin{bmatrix} 16.0 & 12.8 & 11.2 \\ 12.8 & 16.0 & 12.8 \\ 11.2 & 12.8 & 16.0 \end{bmatrix}. \tag{8.45}$$

(a) Write the joint PDF of X_1, X_2 using the algebraic notation of Definition 5.10.
(b) Write the joint PDF of X_1, X_2 using vector notation.
(c) Write the joint PDF of $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$ using vector notation.

. . . .

(a) First we note that X_1 and X_2 have expected values $\mu_1 = 50$ and $\mu_2 = 62$, variances $\sigma_1^2 = \sigma_2^2 = 16$, and covariance $\text{Cov}[X_1, X_2] = 12.8$. It follows from Definition 5.6 that the correlation coefficient is

$$\rho_{X_1,X_2} = \frac{\text{Cov}\left[X_1, X_2\right]}{\sigma_1\sigma_2} = \frac{12.8}{16} = 0.8. \tag{8.46}$$

From Definition 5.10, the joint PDF is

$$f_{X_1,X_2}(x_1,x_2) = \frac{\exp\left[-\dfrac{(x_1-50)^2 - 1.6(x_1-50)(x_2-62) + (x_2-62)^2}{19.2}\right]}{60.3}.$$

(b) Let $\mathbf{W} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$ denote a vector representation for random variables X_1 and X_2. From the covariance matrix $\mathbf{C_X}$, we observe that the 2×2 submatrix in the upper left corner is the covariance matrix of the random vector $\mathbf{W}$. Thus

$$\boldsymbol{\mu}_\mathbf{W} = \begin{bmatrix} 50 \\ 62 \end{bmatrix}, \qquad \mathbf{C_W} = \begin{bmatrix} 16.0 & 12.8 \\ 12.8 & 16.0 \end{bmatrix}. \tag{8.47}$$

We observe that $\det(\mathbf{C_W}) = 92.16$ and $\det(\mathbf{C_W})^{1/2} = 9.6$. From Definition 8.12, the joint PDF of $\mathbf{W}$ is

$$f_\mathbf{W}(\mathbf{w}) = \frac{1}{60.3}\exp\left(-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu}_\mathbf{W})^T\mathbf{C}_\mathbf{W}^{-1}(\mathbf{w}-\boldsymbol{\mu}_\mathbf{W})\right). \tag{8.48}$$

(c) Since $\boldsymbol{\mu}_\mathbf{X} = \begin{bmatrix} 50 & 62 & 58 \end{bmatrix}'$ and $\det(\mathbf{C_X})^{1/2} = 22.717$, $\mathbf{X}$ has PDF

$$f_\mathbf{X}(\mathbf{x}) = \frac{1}{357.8}\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{X})^T\mathbf{C}_\mathbf{X}^{-1}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{X})\right). \tag{8.49}$$