Random Vectors

$\mathbf{X} = \bar{X}$ $\quad$ $\mathbf{x} = \bar{x}$ $\quad$ $'$ means transpose or ^{T}

$$\bar{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$\bar{X}$: Random Vector

$\bar{x}$: Sample Value.

note: $\bar{x}' = [x_1 \; x_2 \; x_3]$

$\bar{x} = [x_1 \; x_2 \; x_3]'$

Notation:

$\mathbb{C}$ $\quad$ $\mathbb{A}$ matrix $\quad$ $\bar{X}$ vector

$\mathbb{R}$

$$F_{\bar{X}}(\bar{x}) = F_{X_1, X_2, X_3, \dots X_n}(x_1, x_2, x_3, \dots, x_4)$$

$\bar{X}$: Random vector $\quad$ $\bar{x}$: sample value.

If $\bar{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ $\qquad$ $\bar{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$

Then: $\bar{W} = \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$

To marginalize e.g $\bar{Y}$ out, you integrate $Y_1\, Y_2\, Y_3$ out

Example 8.1

Random vector $\mathbf{X}$ has PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 6e^{-\mathbf{a}'\mathbf{x}} & \mathbf{x} \geq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{8.1}$$

where $\mathbf{a} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}'$. What is the CDF of $\mathbf{X}$?

What is the dimensions of $\bar{x}$?

$$\underbrace{\bar{a}'_{1\times 3}\ \bar{x}_{?\times 1}}_{3} \qquad a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 + 2x_2 + 3x_3$$

$$f_{\bar{X}}(\bar{x}) = \begin{cases} 6e^{-x_1-2x_2-3x_3} & x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$F_{\bar{X}}(\bar{x}) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_3} f_{\bar{X}}(\bar{u})\, du_3\, du_2\, du_1$$

$$= \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} 6e^{-x_1-2x_2-3x_3}\, du_3\, du_2\, du_1$$

$$= \int_0^{x_1}\int_0^{x_2} 6e^{-x_1-2x_2} \underbrace{\int_0^{x_3} e^{-3u_3}\, du_3}\, du_2\, du_1$$

$$-\frac{1}{3}e^{-3u_3}\Big|_0^{x_3}$$

$$6\left(-e^{-x_1}+1\right)\left(-\frac{1}{2}e^{-2x_2}+\frac{1}{2}\right)\left(-\frac{1}{3}e^{-3x_3}+\frac{1}{3}\right)$$

$$F_{\bar{x}}(\bar{x}) = \begin{cases} (1-e^{-x_1})(1-e^{-2x_2})(1-e^{-3x_3}) & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 8.2

As in Example 5.22, random variables $Y_1, \ldots, Y_4$ have the joint PDF

$$f_{\bar{Y}}(\bar{y}) = \underbrace{f_{Y_1,\ldots,Y_4}(y_1,\ldots,y_4)}_{f_{\bar{Y}}(\bar{y})} = \begin{cases} 4 & 0 \le y_1 \le y_2 \le 1, 0 \le y_3 \le y_4 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.4}$$

Let $\mathbf{V} = \begin{bmatrix} Y_1 & Y_4 \end{bmatrix}'$ and $\mathbf{W} = \begin{bmatrix} Y_2 & Y_3 \end{bmatrix}'$. Are $\mathbf{V}$ and $\mathbf{W}$ independent random vectors?

$$\bar{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_4 \end{bmatrix} \qquad \bar{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} Y_2 \\ Y_3 \end{bmatrix}$$

$$f_{\bar{V},\bar{W}}(\bar{v},\bar{w}) = f_{V_1,V_2,W_1,W_2}(v_1, v_2, w_1, w_2) = \begin{cases} 4 & 0 \le v_1 \le w_1 \le 1 \\ & 0 \le w_2 \le v_2 \le 1 \\ 0 & \text{otherw} \end{cases}$$

$$\left(= f_{Y_1,Y_4,Y_2,Y_3}(y_1, y_4, y_2, y_3)\right)$$

$$\int_0^{v_2}\int_{v_1}^{1} 4 \, dw_1 \, dw_2 \quad f_{\bar{v}}(\bar{v})$$

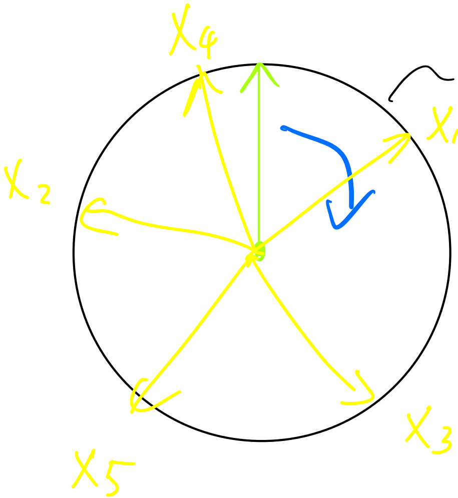
$$f_{\bar{v}}(\bar{v}) = \begin{cases} (4-4v_1)\,v_2 & 0 \le v_1 \le 1 \quad 0 \le v_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

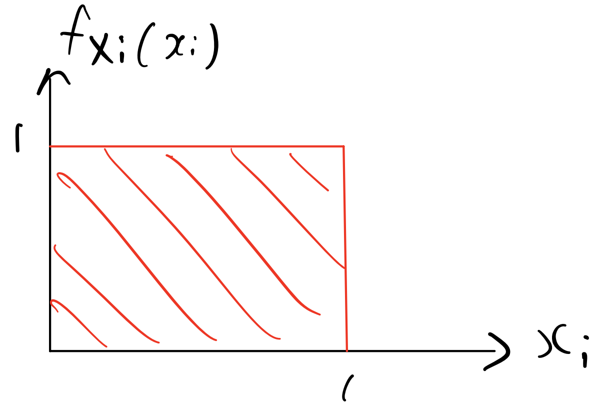
$$f_{\bar{w}}(\bar{w}) = \int_{w_2}^{1}\int_0^{w_1} 4 \, dv_1 \, dv_2$$

$$= \begin{cases} 4w_1(1-w_2) & 0 \le w_1 \le 1 \quad 0 \le w_2 \le 1 \\ 0 & \text{otherw} \end{cases}$$

$$f_{\bar{v},\bar{w}}(\bar{v},\bar{w}) \stackrel{?}{=} f_{\bar{v}}(\bar{v}) f_{\bar{w}}(\bar{w}) \quad \times \quad \text{No, not indep.}$$

Example 8.3





X_i is position of pointer

$$Y_n = \max(X_1, X_2, X_3, \cdots, X_n) = g(\bar{X})$$

$$F_{Y_n}(y) = P(Y_n \le y) =$$

$$= P(\max(X_1, X_2, X_3, \cdots, X_n) \le y) =$$

$\boxed{F_X(x)} = P(X \le x)$

$$= P(X_1 \le y, X_2 \le y, \ldots, X_n \le y)$$

$$= P(X_1 \le y) \cdot P(X_2 \le y) \cdot \ldots P(X_n \le y)$$

$$= [F_X(y)]^n$$

i.i.d.

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > \end{cases}$$

$$F_{Y_n}(y) = \begin{cases} 0 & y < 0 \\ y^n & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases} \rightarrow \frac{d}{dy} = ,$$

$$f_{Y_n}(y) = \begin{cases} n\, y^{n-1} & 0 \le y < 1 \\ 0 & \text{otherwise} \end{cases}$$

From Lecture 18:

$W = aX \Rightarrow$

$$F_W(w) = F_X\left(\frac{w}{a}\right)$$

$$f_W(w) = \frac{1}{a} f_X\left(\frac{w}{a}\right)$$

$W = X + b \Rightarrow$

$$F_W(w) = F_X(w-b)$$

$$f_W(w) = f_X(w-b)$$

$$\bar{Y} = A\bar{x} + \bar{b}$$

$$\bar{b} = [0 \quad 0 \quad 0]'$$

$$\underset{3 \times 3}{\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}} \cdot \underset{3 \times 1}{\overset{\bar{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}} = \overset{\bar{Y}}{\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 + 8x_2 + 9x_3 \end{bmatrix}} \begin{array}{l} = y_1 \\ = y_2 \\ = y_3 \end{array}$$