Conditioning by a random Variable

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Last time

$$P_{X|B}(x) = P_{X|X>3}(x) \qquad \text{one RV}$$

$$P_{X,Y|B}(x,y) = P_{X,Y|X+Y>2}(x,y) \qquad \text{two RVs}$$

$$= \frac{P_{X,Y}(x,y)}{P(B)}$$

$B: Y=y$

Today

$$P_{X|Y}(x|y) = ? \qquad \text{two RVs}$$

$$P(\overset{A}{X=x} \mid \overset{B}{Y=y}) = \frac{P(\overset{A}{X=x} \cap \overset{B}{Y=y})}{P(Y=y)}$$

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{\overset{B}{P_Y(y)}} = P(B)$$

where $P_{X,Y}(x,y)$ exist.

from:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

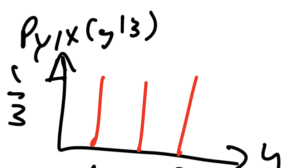
$$P(A \cap B) = P(A|B)\,P(B) = P(B|A)\,P(A)$$

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\,f_Y(y)$$
$$= f_{Y|X}(y|x) \cdot f_X(x).$$

if independent $f_{Y|X}(y|x) = f_Y(y)$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)\,\cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)$$

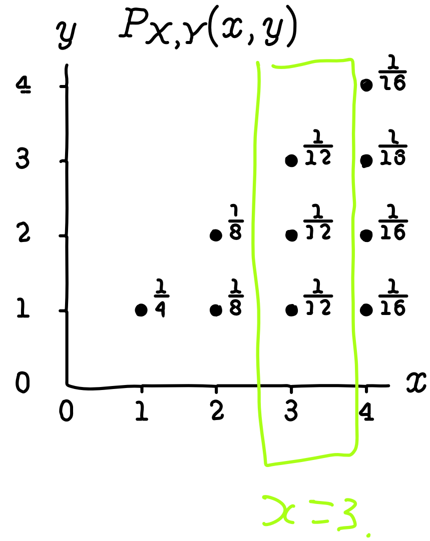


Example 7.12

$P_{Y|X}(y|3) = ?$

$x = 3$

$$\boxed{P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}}$$



$$P_{Y|X}(y|3) = \begin{cases} \frac{1}{3} & y = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x,y)$$

$$= \begin{cases} \frac{1}{4} & x = 1 \\ \frac{1}{4} & x = 2 \\ \frac{1}{4} & x = 3 \\ \frac{1}{4} & x = 4 \end{cases} = \begin{cases} \frac{1}{4} & x = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)} = 4\, P_{X,Y}(x,y)$$

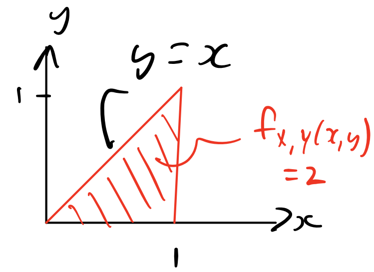
$x = 3$

$$= 4 \cdot \frac{1}{12} \quad y = 1, 2, 3$$

$$= \begin{cases} \frac{1}{3} & y = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

Example 7.13

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$



• For $0 \le x \le 1$ find $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{1}{x} & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$f_X(x) = \int_0^x 2\, dy.$$
$$= 2x$$

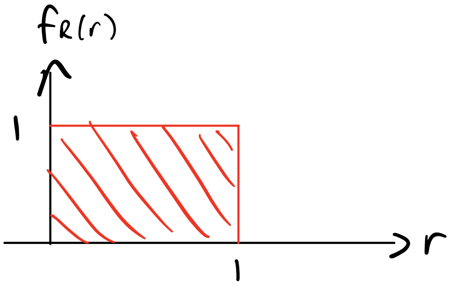
• For $0 \le y \le 1$ find $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{1-y} & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

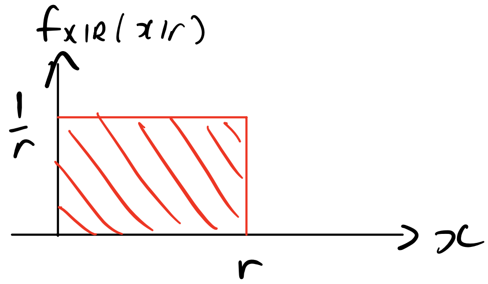
$$f_Y(y) = \int_y^1 2\, dx = 2 - 2y$$

Example 7.14

$$f_R(r) = \begin{cases} 1 & 0 \le r < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|R}(x|r) = \begin{cases} \frac{1}{r} & 0 \le x \le r \\ 0 & \text{otherwise} \end{cases}$$





$$f_{R|X}(r|x) = ? = \frac{f_{X|R}(x|r)\, f_R(r)}{f_X(x)}$$

$$f_{R|X}(r|x) = \frac{f_{R,X}(r,x)}{f_X(x)} \quad \begin{matrix} ① \\ ② \end{matrix} = \begin{cases} \dfrac{1}{-r \ln x} & 0 \le x \le r < 1 \\ 0 & \text{otherwise} \end{cases}$$

① $$f_{R,X}(r,x) = f_{X|R}(x|r) \cdot f_R(r)$$

$$= \begin{cases} \frac{1}{r} & 0 \le x \le r < 1 \\ 0 & \text{otherwise} \end{cases}$$

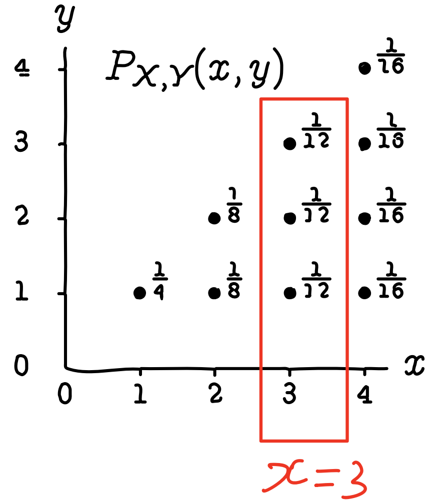
②

$$f_X(x) = \int_x^1 \frac{1}{r}\, dr$$

$$= \ln r \Big|_x^1 = 0 - \ln x = -\ln x$$

Example 7.15

$$P_{Y|X}(y|3) = \begin{cases} \frac{1}{3} & y = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$



$$E[Y \mid X=3] = \sum_{y \in S_Y} y \cdot P_{Y|X}(y|3)$$

$$= (1+2+3)\frac{1}{3}$$

$$= 2$$

Example 7.16

$E[X \mid Y=y]$ and $\overbrace{E[X \mid Y]}^{RV}$ — What is the difference?

$\underbrace{E[X \mid Y=y]}_{\text{function of } y}$

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\, dx$$

$$= \int_y^1 \frac{x}{1-y}\, dx = \frac{1+y}{2} \qquad 0 \le y \le 1$$

$$E[X \mid Y=y] = \frac{1+y}{2}$$ ← value.

$$E[X \mid Y] = \frac{1+Y}{2}$$ ← RV

$$E[\underbrace{E[X \mid Y]}_{RV.}] = E[X]$$

Iterated expectation

$$E[E[X|Y]] = E[X] \qquad E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

Proof:

$$E[\underbrace{E[X|Y]}_{g(Y)}] = \int_{-\infty}^{\infty} \underbrace{E[X|Y=y]}_{g(y)}\, f_Y(y)\,dy.$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx \right) \cdot f_Y(y)\,dy.$$

$$= \int_{-\infty}^{\infty} x \underbrace{\left(\int_{-\infty}^{\infty} \underbrace{f_{X|Y}(x|y)\, f_Y(y)}_{f_{X,Y}(x,y)}\,dy \right)}_{\text{Marginalization} \rightarrow f_X(x)} dx$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$= E[X]$$