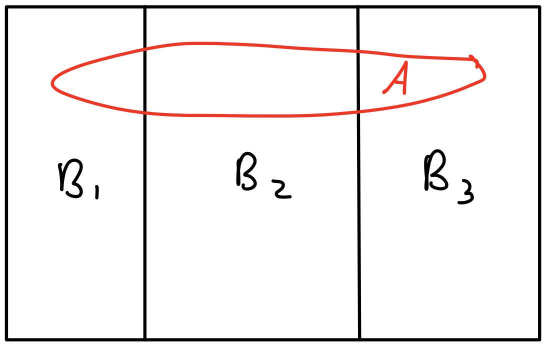
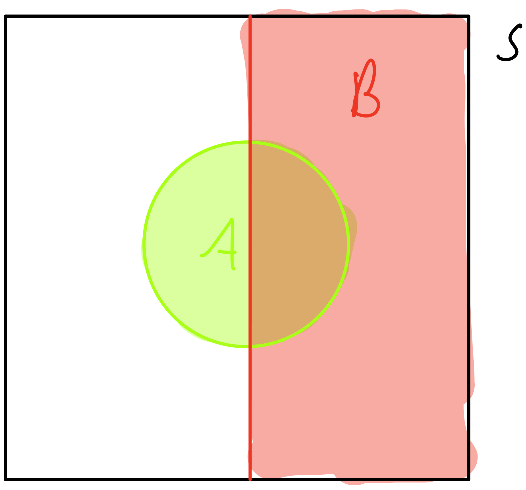
Conditional Probability Models



Total Probability:

$$P(A) = \sum_{i=1}^{3} P(A \mid B_i) \cdot P(B_i)$$



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Previously:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

$$F_X(x) = P(X \leq x)$$

$$P(A \cap B) = P(A \mid B)\, P(B)$$
$$= P(B \mid A)\, P(A)$$

A A A

$$F_{\underset{\text{RV}}{X}|\underset{\text{Event}}{B}}(x) = \frac{P(X \le x \cap B)}{P(B)} = \frac{P(B| X \le x)\cdot P(X \le x)}{P(B)}$$

$$= \frac{P(B| X \le x)\cdot F_X(x)}{P(B)}$$

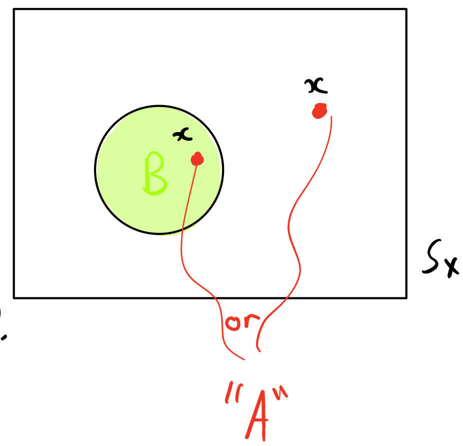
(Discrete or Continuous)

Discrete Case

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$x \in B$: $\underbrace{\{X=x\}}_{A} \cap B = \underbrace{\{X=x\}}_{A}$

$$P(X=x \cap B) = P(X=x) = P_X(x)$$



$x \notin B$ $\{X=x\} \cap B = \phi$

$$P(X=x \cap B) = 0$$

$$P_{X|B}(x) = P(X=x|B) = \begin{cases} \dfrac{P_X(x)}{P(B)} & x \in B \\ 0 & x \notin B \end{cases}$$

Example 7.2

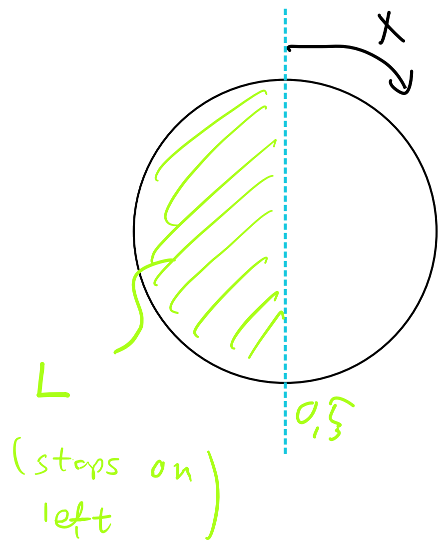
$$P_X(x) = \begin{cases} 0.15 & x = 1,2,3,4, \leftarrow \text{Server 1} - \text{short } (S) \\ 0.1 & x = 5,6,7,8, \leftarrow \text{Server 2} - \text{long } (L) \\ 0 & \text{otherwise.} \end{cases}$$

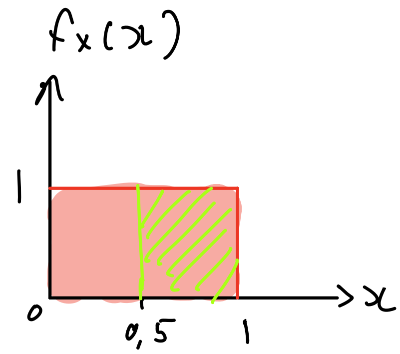
$$P_{X|L}(x) = \begin{cases} \dfrac{P_X(x)}{P(L)} & x \in L \\ 0 & x \notin L \end{cases}$$

$$P(L) = \sum_{x=5}^{8} P_X(x) = 0{,}4 \longrightarrow$$

$$P_{X|L}(x) = \begin{cases} \dfrac{0{,}1}{0{,}4} = 0{,}25 & x = 5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

Example 7.3





$$f_{X|L}(x) = \begin{cases} \dfrac{f_X(x)}{P(L)} & x \in L \\ 0 & \text{otherwise} \end{cases}$$

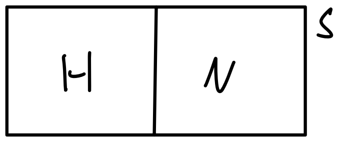
$$P(L) = \frac{1}{2}$$

$$= \begin{cases} 2 & 0,5 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 7.6

Let X denote the number of additional years that a randomly chosen 70-year-old person will live. If the person has high blood pressure, denoted as event H, then X is a geometric $(p = 0.1)$ random variable. Otherwise, if the person's blood pressure is normal, event N, X has a geometric $(p = 0.05)$ PMF. Find the conditional PMFs $P_{X|H}(x)$ and $P_{X|N}(x)$. If 40 percent of all 70-year-olds have high blood pressure, what is the PMF of X?

Example 7.6



num years still to live.

H : high blood pressure $\longrightarrow$ X : Geometric $(p = 0{,}1)$

N : Normal blood pressure $\longrightarrow$ X : Geometric $(p = 0{,}05)$

$p(1-p)^{x-1}$

$P(H) = 0{,}4$ $\qquad P(N) = 1 - P(H)$
$\qquad\qquad\qquad\qquad = 0{,}6$

$$P_{X|H}(x) = \begin{cases} 0{,}1(0{,}9)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$P_{X|N}(x) = \begin{cases} 0{,}05(0{,}95)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$P_X(x) = P_{X|H}(x)\,P(H) + P_{X|N}(x) \cdot P(N)$$

$$= \begin{cases} 0{,}1(0{,}9)^{x-1} \cdot 0{,}4 + 0{,}05(0{,}95)^{x-1} \cdot 0{,}6 & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Proof

$$E[X] = \sum_{x \in S_X} x P_X(x)$$

$$= \sum_{x \in S_X} x \cdot \left[\sum_{i=1}^{m} P_{X|B_i}(x) P(B_i) \right]$$ — Total Probability.

$$= \sum_{i=1}^{m} P(B_i) \sum_{x \in S_X} x \cdot P_{X|B_i}(x)$$

$$= \sum_{i=1}^{m} P(B_i) E[X|B_i]$$

Example 7.10

X and Y are random variables with joint PDF

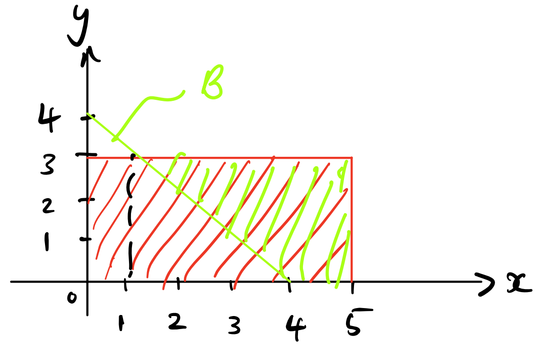
$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 5, 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases} \tag{7.22}$$

Find the conditional PDF of X and Y given the event $B = \{X + Y \ge 4\}$.

Example 7.10

$$P(B) = \int_0^3 \int_{4-y}^5 \frac{1}{15} \, dx \, dy.$$

$$= \frac{1}{2}$$



$$B = \{ X + Y \geq 4 \}$$

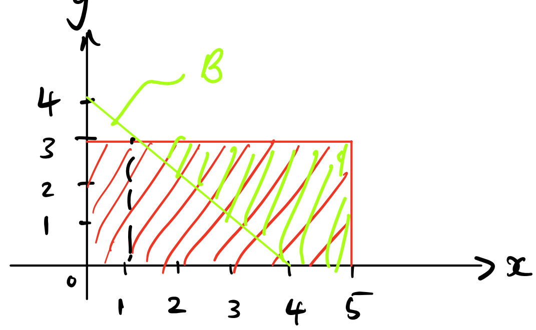
$$= \{ Y \geq -X + 4 \}$$

$$= \{ X \geq 4 - Y \}$$

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P(B)} & (x,y) \in B \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{2}{15} & x + y \geq 4 \\ & 1 \leq x \leq 5, \quad 0 \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Example



$W = XY \qquad B = \{X+Y \geq 4\}$

$$E[W|B] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y|B}(x,y)\,dx\,dy.$$

$$= \int_0^3 \int_{-y+4}^{5} xy \frac{2}{15}\,dx\,dy.$$

$$= \frac{2}{15}\int_0^3 y \int_{-y+4}^{5} x\,dx\,dy.$$

$$\vdots$$

$$= \underline{6{,}15}$$

$$E[W^2|B] = \int_0^3 \int_{-y+4}^{5} (xy)^2 \frac{2}{15}\,dx\,dy.$$

$$\vdots$$

$$Var[W|B] = E[W^2|B] - E[W|B]^2$$