$$P = \frac{V^2}{r}$$

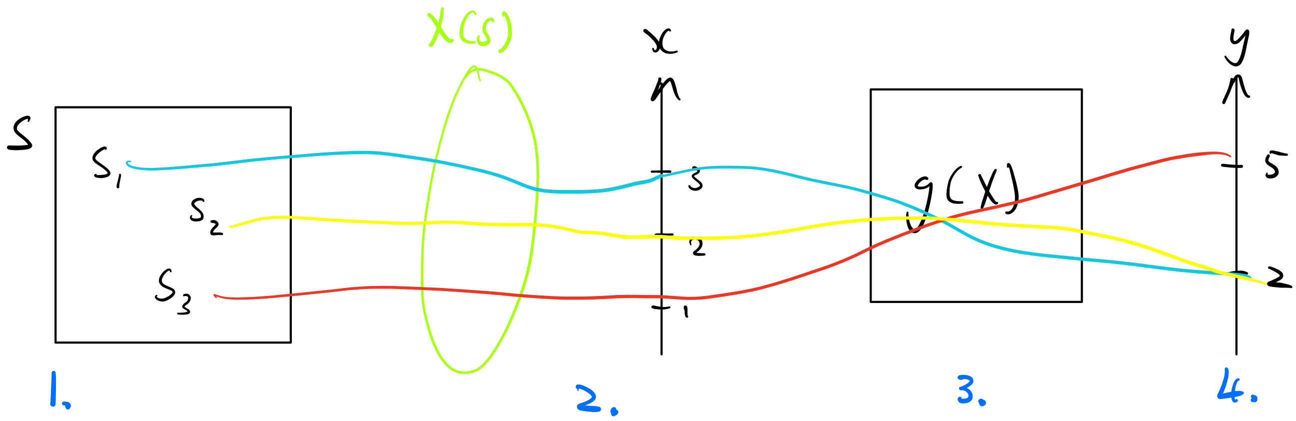
$$W = g(X)$$

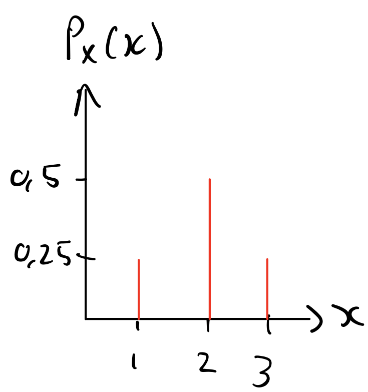
Then

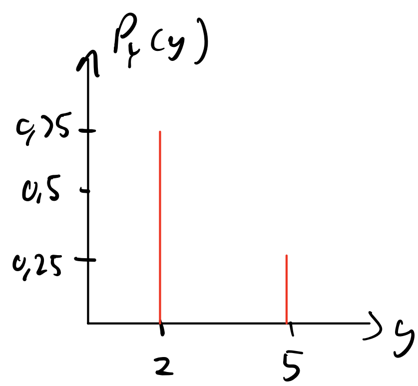
$$E[W] = E[g(X)] = \sum_{x \in S_X} g(x)\, P_X(x)$$

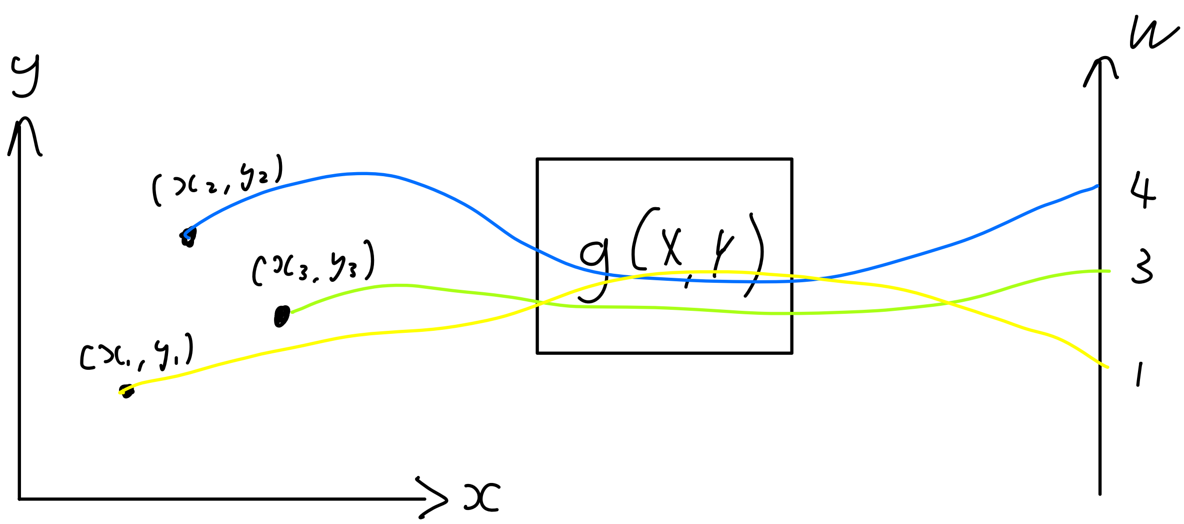
$$= \sum_{w \in S_W} w \cdot P_W(w)$$

PMF Steps (Discrete)









Example 6.1

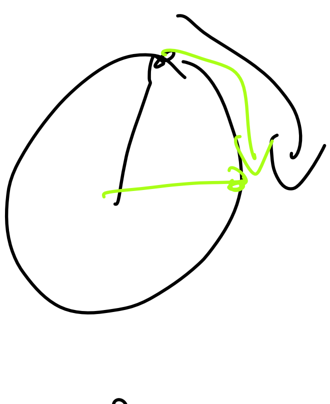
		Cost(X)	
		Grayscale	Colour
	$P_{L,X}(l,x)$	$x = 40$	$x = 60$
Length (L)	$l = 1$	0.15 ($w=40$)	0.1 ($w=60$)
	$l = 2$	0.3 ($w=80$)	0.2 ($w=120$)
	$l = 3$	0.15 ($w=120$)	0.1 ($w=180$)

Now, $W = LX = g(L, X)$

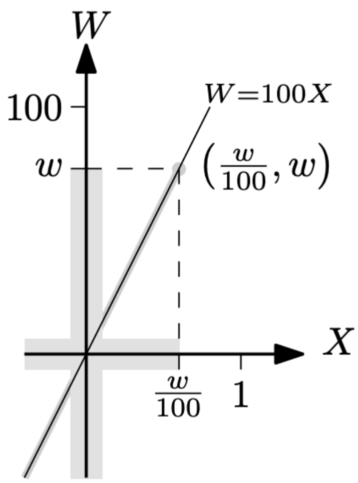
w	40	60	80	120	180
$P_W(w)$	0,15	0,1	0,3	0,35	0,1

$$S_W = \{40, 60, 80, 120, 180\}$$

$$F_W(w) = P(W \leq w) = P(g(X) \leq w)$$
$$= P(X \leq g^{-1}(w))$$
$$= F_X(g^{-1}(w))$$



$$\frac{\partial(100x)}{\partial x} = 100 \qquad W = g(X) = 100X \qquad X = g^{-1}(w) = \frac{w}{100}$$



The function $\boxed{W = 100X,}$ where X in Example 4.2 is the location of the pointer measured in meters. To find the CDF $F_W(w) = \mathrm{P}[W \le w]$, the first step is to translate the event $\{W \le w\}$ into an event described by X. Each outcome of the experiment is mapped to an (X, W) pair on the line $W = 100X$. Thus the event $\{W \le w\}$, shown with gray highlight on the vertical axis, is the same event as $\{X \le w/100\}$, which is shown with gray highlight on the horizontal axis. Both of these events correspond in the figure to observing an (X, W) pair along the highlighted section of the line $w = g(X) = 100w$. This translation of the event $W = w$ to an event described in terms of X depends only on the function $g(X)$. Specifically, it does not depend on the probability model for X. From the figure, we see that

$$F_W(w) = \mathrm{P}[W \le w] = \mathrm{P}[100X \le w] = \mathrm{P}[X \le w/100] = F_X(w/100). \tag{6.1}$$

The calculation of $F_X(w/100)$ depends on the probability model for X. For this problem, we recall that Example 4.2 derives the CDF of X,

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x & 0 \le x < 1, \\ 1 & x \ge 1. \end{cases} \tag{6.2}$$

Uniform distribution

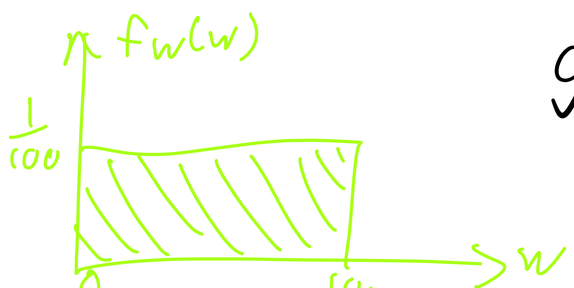
$$f_X(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

From this result, we can use algebra to find

$$F_W(w) = F_X\left(\frac{w}{100}\right) = \begin{cases} 0 & \frac{w}{100} < 0, \\ \frac{w}{100} & 0 \le \frac{w}{100} < 1, \\ 1 & \frac{w}{100} \ge 1, \end{cases} = \begin{cases} 0 & w < 0, \\ \frac{w}{100} & 0 \le w < 100, \\ 1 & w \ge 100. \end{cases} \tag{6.3}$$

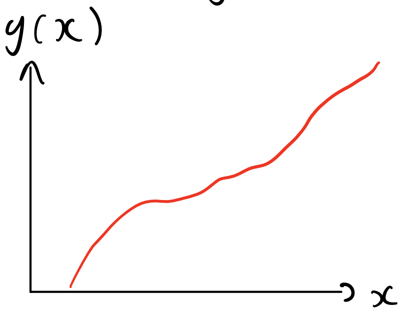
We take the derivative of the CDF of W over each of the intervals to find the PDF:

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 1/100 & 0 \le w < 100, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

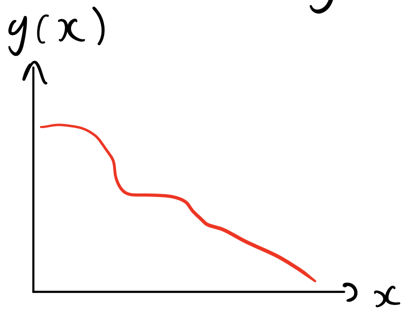


$$g^{-1}(w) = \frac{w}{100}$$

Monotonically Increasing



Monotonically Decreasing



Proofs $\quad$ get $f_W(w)$ in terms of $f_X(x)$

- $W = aX \qquad a > 0$

2 Steps: [1. CDF $\quad$ 2. $\frac{d}{dw}$] $\qquad g(X) = aX$

1. $F_W(w) = P(W \le w) = P(aX \le w) = P(X \le \frac{w}{a}) = F_X(\frac{w}{a})$

2. $f_W(w) = \frac{dF_W(w)}{dw} = \frac{dF_X(\frac{w}{a})}{dw} = \frac{1}{a} f_X(\frac{w}{a})$

e.g. $F_X(x) = x^3 + 5 \qquad f_X(x) = \frac{dF_X(x)}{dx} = 3x^2$

$F_W(w) = F_X(\frac{w}{a}) = (\frac{w}{a})^3 + 5 = \frac{w^3}{a^3} + 5$

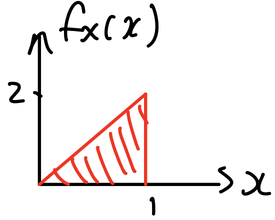
$f_W(w) = \frac{dF_W(w)}{dw} = \frac{3}{a^3} w^2 = \frac{3}{a}\left(\frac{w}{a}\right)^2 = \frac{1}{a} f_X\left(\frac{w}{a}\right)$

- $W = X + b$

$F_W(w) = P(X + b \le w) = P(X \le w - b) = F_X(w - b)$

$\frac{dF_X(w-b)}{dw} = \ldots = f_X(w-b)$

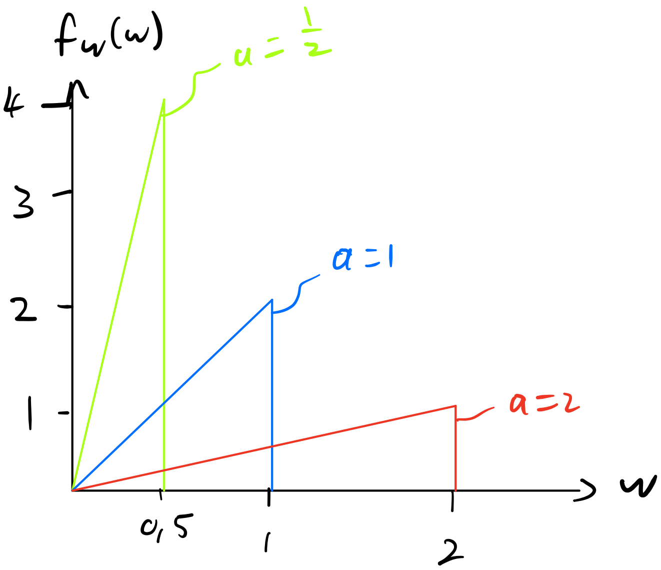
Example 6.3



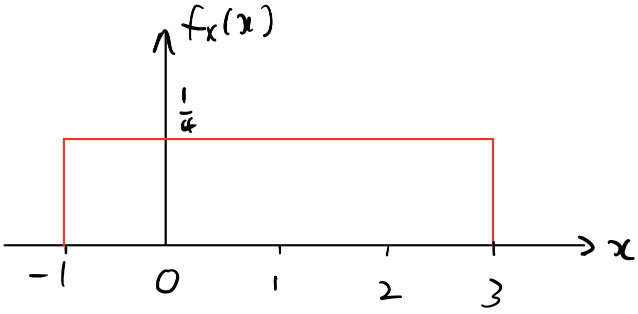
$W = aX$

$$\Rightarrow f_W(w) = \frac{1}{a} f_X\left(\frac{w}{a}\right) \Rightarrow \frac{1}{a}\left[2\left(\frac{w}{a}\right)\right] = \frac{2w}{a^2} \qquad 0 \le \frac{w}{a} \le 1$$

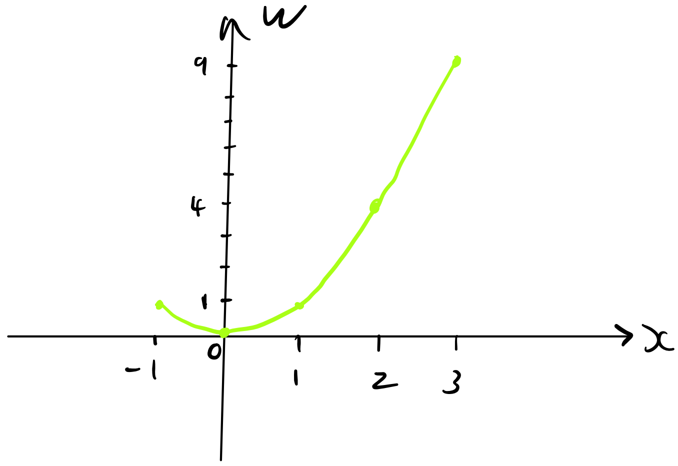
$$f_W(w) = \begin{cases} \dfrac{2w}{a^2} & 0 \le w \le a \\ 0 & \text{otherwise} \end{cases}$$



Example



$$f_X(x) = \begin{cases} \frac{1}{4} & -1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$



$$W = X^2 = g(X)$$

range: $-1 < x < 3$

$\Rightarrow 0 < W < 9$?

double up at $0 < W < 1$

Approach 1 : CDF $\Rightarrow$ PDF

1. Find CDF of $F_W(w)$:

$$F_W(w) = P(W \leq w) = P(X^2 \leq w)$$

$$= P(-\sqrt{w} \leq X \leq \sqrt{w})$$

$$= F_X(\sqrt{w}) - F_X(\sqrt{w})$$

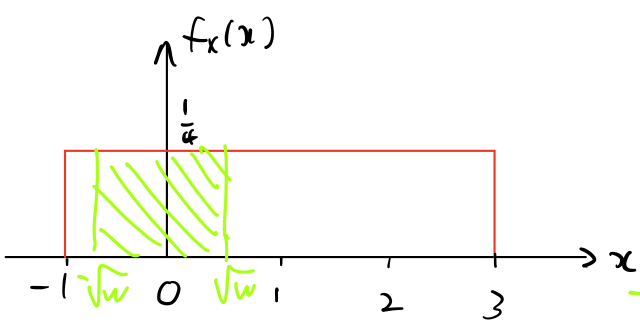
$$= \begin{cases} \int_{-\sqrt{w}}^{\sqrt{w}} f_X(x)\,dx & \text{only } 0 \leq w \leq 1 \\ \int_{-1}^{\sqrt{w}} f_X(x)\,dx & \text{only } 1 \leq w \leq 9 \end{cases}$$

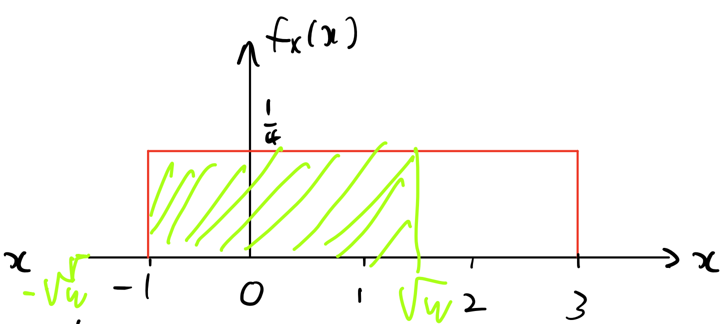
$X^2 = w$

$X = \pm\sqrt{w}$

$0 \leq w \leq 1$

$1 \leq w \leq 9$





$$F_W(w) = \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{4}\,dx$$

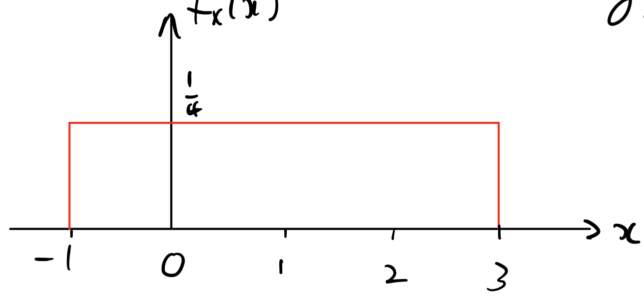
$$F_W(w) = \int_{-1}^{\sqrt{w}} \frac{1}{4}\,dx$$

$= \frac{\sqrt{w}}{2}$

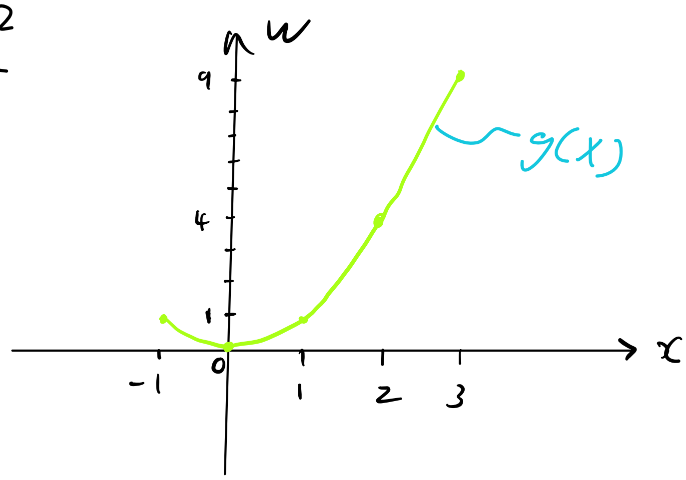
$= \frac{\sqrt{w}+1}{4}$

$$F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{\sqrt{w}}{2} & 0 \le w \le 1 \\ \frac{\sqrt{w}+1}{4} & 1 < w \le 9 \\ 1 & w > 9 \end{cases} \xrightarrow{\frac{d}{dw}} f_W(w) = \begin{cases} \frac{1}{4\sqrt{w}} & 0 \le w \le 1 \\ \frac{1}{8\sqrt{w}} & 1 \le w \le 9 \\ 0 & \text{otherwise} \end{cases}$$

Example



$$\frac{\partial g(x)}{\partial x} = \frac{\partial x^2}{\partial x} = 2x$$



$$f_x(x) = \begin{cases} \frac{1}{4} & -1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{W = X^2} \Rightarrow x = g^{-1}(w) = \pm\sqrt{w}$$

Approach 2: If $g(x)$ monotonically increasing or decreasing

$$f_w(w) = \frac{f_x(x)}{\left|\frac{\partial g(x)}{\partial x}\right|}\Bigg|_{x=\pm\sqrt{w}}$$

Decreasing $\begin{pmatrix} -1 \le x \le 0 \\ 0 \le w \le 1 \end{pmatrix}$ Increasing $\begin{pmatrix} 0 \le x \le 3 \\ 0 \le w \le 9 \end{pmatrix}$

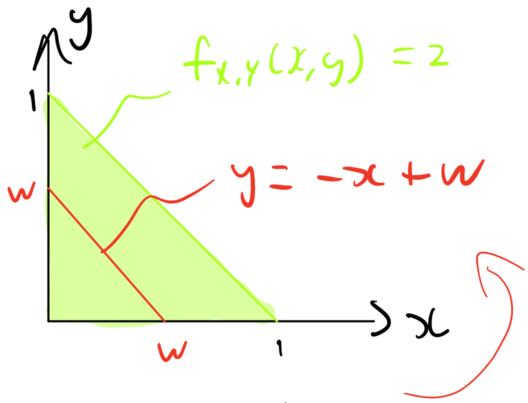
$x = -\sqrt{w}$ applicable $x = +\sqrt{w}$ applicable.

$$\frac{f_x(x)}{|2(-\sqrt{w})|}\Bigg|_{\text{for } 0 \le w \le 1} + \frac{f_x(x)}{|2(\sqrt{w})|}\Bigg|_{\text{for } 0 \le w \le 9}$$

$$= \frac{1}{8\sqrt{w}}\Bigg|_{0 \le w \le 1} + \frac{1}{8\sqrt{w}}\Bigg|_{0 \le w \le 9}$$

$$f_W(w) = \begin{cases} \dfrac{1}{4\sqrt{w}} & 0 \leq w \leq 1 \\ \dfrac{1}{8\sqrt{w}} & 1 \leq w \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

Example 6.11



$W = X + Y$

$$f_W(w) = \int_{-\infty}^{\infty} f_{XY}(x, w-x)\,dx$$

$$=$$