Example 5.19

$$Y = \overbrace{X + Z}^{\text{Independent}}$$

$$X = \text{Gaus}(0, \sigma_x)$$
$$Z = \text{Gaus}(0, \sigma_z)$$

$\rho_{x,z} = 0$ (since independent)

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]\,Var[Y]}}$$

$$= \frac{\sigma_x^2}{\sqrt{\sigma_x^2(\sigma_x^2 + \sigma_z^2)}} \qquad \times \sqrt{\frac{\frac{1}{\sigma_z^2}}{\frac{1}{\sigma_z^2}}}$$

$$= \sqrt{\frac{\left(\frac{\sigma_x}{\sigma_z}\right)^2}{1 + \left(\frac{\sigma_x}{\sigma_z}\right)^2}}$$

SNR.

What is $f_Y(y)$?

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

$$\mu_Y = E[Y] = E[X] + E[Z] = 0$$

$$\sigma_Y^2 = Var[Y] = Var[X] + Var[Z] + 2\,Cov[X,Z]$$
$$= \sigma_x^2 + \sigma_z^2$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}(\sigma_x^2 + \sigma_z^2)} e^{-\frac{y^2}{2(\sigma_x^2 + \sigma_y^2)^2}}$$

Example 5.20

$f_{X,Y}(x,y)$ $\qquad \sigma_X = 4 \qquad \sigma_Z = 3$

$$f_{X,Y}(x,y) = \frac{e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\left(\frac{x-\cancel{\mu_X}}{\sigma_X}\right)^2 - \frac{2\rho_{X,Y}(x-\cancel{\mu_X})(y-\cancel{\mu_Y})}{\sigma_X\sigma_Y} + \left(\frac{y-\cancel{\mu_Y}}{\sigma_Y}\right)^2\right]}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$

Need: μ_X ✓, μ_Y ✓, σ_X ✓, σ_Y ✓, ρ_{XY}

(last lecture:)

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]\,Var[Y]}}$$

$$= \sqrt{\frac{\left(\frac{\sigma_X}{\sigma_Z}\right)^2}{1 + \left(\frac{\sigma_X}{\sigma_Z}\right)^2}}$$

SNR.

$$= \frac{4}{5} = 0.8$$

$$\sigma_Y = \sqrt{\sigma_X^2 + \sigma_Z^2}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25} = 5$$

$$f_{X,Y}(x,y) = \frac{1}{24\pi} e^{-\left(\frac{25}{16}x^2 - 2xy + y^2\right)/18}$$

Example 5.22

$$f_{Y_1,\dots,Y_4}(y_1,\dots,y_4) = \begin{cases} 4 & 0 \le y_1 \le \cancel{y_2} \le 1, \quad 0 \le \cancel{y_3} \le y_4 \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{Y_1,Y_4}(y_1, y_4) = \int_{y_1}^{1} \int_{0}^{y_4} 4 \, dy_3 \, dy_2$$

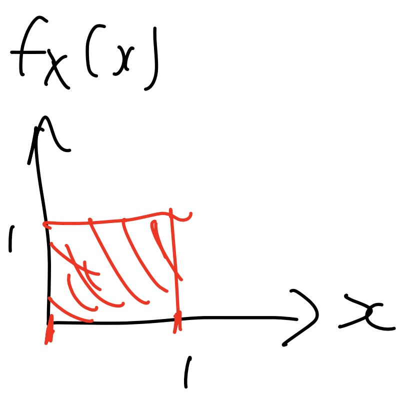
$$= \int_{y_1}^{1} 4 y_3 \Big|_0^{y_4} dy_2$$

$$= \int_{y_1}^{1} 4 y_4 \, dy_2 = 4 y_4 y_2 \Big|_{y_1}^{1}$$

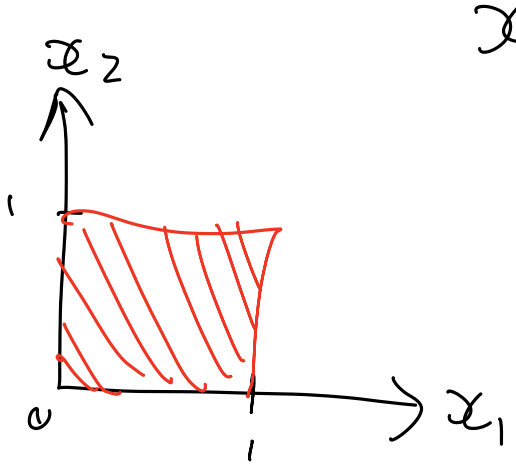
$$= \begin{cases} 4 y_4 (1 - y_1) & 0 \le y_1 \le 1 \\ & 0 \le y_4 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y_1}(y_1) = \int_0^1 4 y_4 (1 - y_1) \, dy_4$$

$$= \begin{cases} 2(1 - y_1) & 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

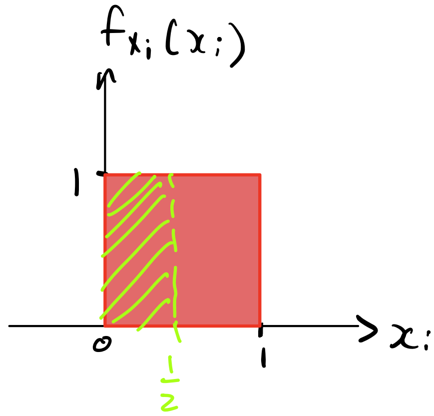


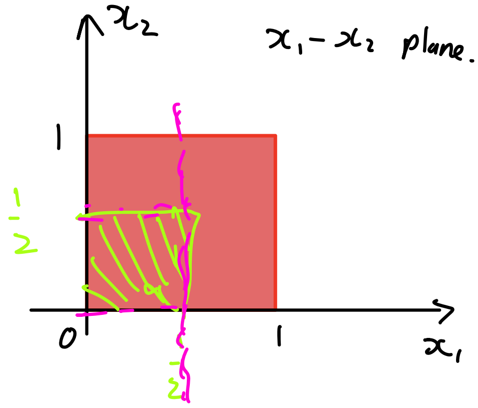
$x_1 - x_2$ plane



Example 5.23

$A := \max_i X_i \le \frac{1}{2}$





for $n=1$

$P(A) = \frac{1}{2}$

for $n=2$

$P(A) = \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$

General:

$$P(A) = P\left(\max_i X_i \le \frac{1}{2}\right) = P\left(X_1 \le \frac{1}{2} \wedge X_2 \le \frac{1}{2} \cdots \wedge X_n \le \frac{1}{2}\right)$$

$$= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \cdots \int_0^{\frac{1}{2}} 1 \; dx_1 \, dx_2 \ldots dx_n$$

$$x_1 \Big|_0^{\frac{1}{2}}$$

$$\frac{1}{2} \, x_2 \Big|_0^{\frac{1}{2}}$$

$$\frac{1}{4}$$

$$\vdots$$

$$= \frac{1}{2^n}$$

$$P_{X_i}\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$P_{X_1}\left(\frac{1}{2}\right) \cdot P_{X_2}\left(\frac{1}{2}\right) \cdots P_{X_n}\left(\frac{1}{2}\right) \ldots = \left[P_{X_i}\left(\frac{1}{2}\right)\right]^n$$

Clearly $X_1, \cdots, X_n$ are iid $PDF = \text{Cont. Uniform}(0,1)$

$$P(A) = P(X_1 \leq \tfrac{1}{2}) \cdot P(X_2 \leq \tfrac{1}{2}) \cdots P(X_n \leq \tfrac{1}{2})$$
$$= \left(\tfrac{1}{2}\right)^n$$

if
iid: $P_{X_1 \dots X_n}(x_1 \dots x_n) =$