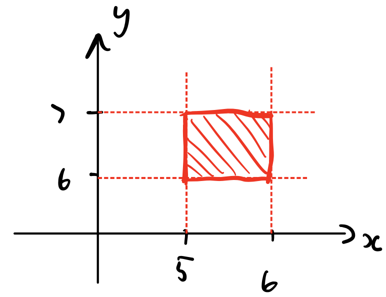
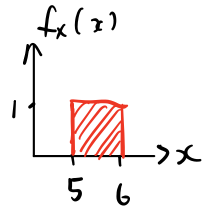
Independence Examples

1. Children's ages.

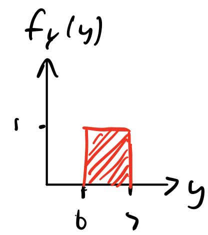
$$f_{X,Y}(x,y) = \begin{cases} 1 & 5 \le x < 6, \quad 6 \le y < 7 \\ 0 & \text{otherwise} \end{cases}$$



$$f_X(x) = \begin{cases} 1 & 5 \le x < 6 \\ 0 & \text{otherwise} \end{cases}$$



$$f_Y(y) = \begin{cases} 1 & 6 \le y < 7 \\ 0 & \text{otherwise} \end{cases}$$



Independence?

$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$? Yes, independent.

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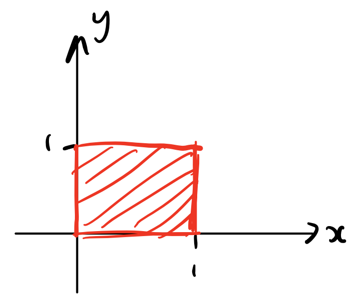
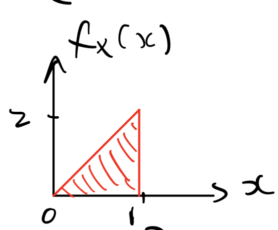
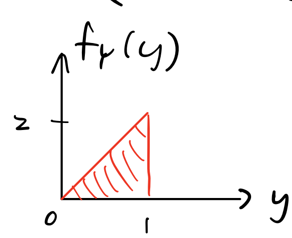
2. $$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

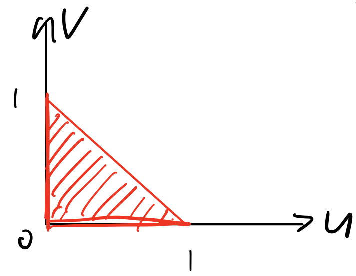


Independence?

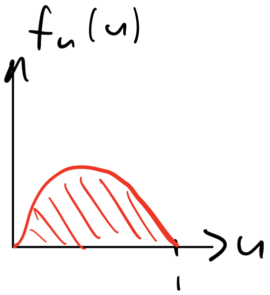
$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$? Yes, independent.

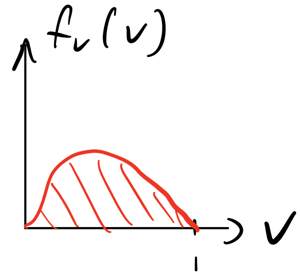
3.

$$f_{UV}(u,v) = \begin{cases} 24uv & u \geq 0, \quad v \geq 0, \quad u+v \leq 1 \\ & (\text{or } v \leq 1-u) \\ 0 & \text{otherwise} \end{cases}$$



$$f_U(u) = \begin{cases} 12u(1-u)^2 & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_V(v) = \begin{cases} 12v(1-v)^2 & 0 \leq v \leq 1 \\ 0 & \text{otherwise} \end{cases}$$





Independence?

$f_{UV}(u,v) = f_U(u) \cdot f_V(v)$? No, not independent.

Derived Random Variable with 1 RV

$Y = g(X) = 3X + 5$

$$E[\underbrace{g(X)}_{Y}] = \sum_{x \in S_X} g(x) P_X(x)$$

$= E[Y]$

Derived Random Variables

$$P_{X,Y}(x,y) = \begin{cases} \frac{x+y}{32} & \text{for } x = 1,2 \text{ and } y = 1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

$$Z = 3X + 4Y$$

Calculate $E[Z]$:

1. $E[Z] = 3E[X] + 4E[Y]$

2. $E[Z] = \sum_{z \in S_Z} z \cdot P_Z(z)$

3. $E[g(X,Y)] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y) \, P_{X,Y}(x,y)$

$$= \sum_{x \in S_X} \sum_{y \in S_Y} (3x + 4y) \cdot \left(\frac{x+y}{32}\right)$$

Set $Z = X + Y$: $\Rightarrow \mu_Z = \mu_X + \mu_Y$

$$Var[X+Y] = Var[Z] = E[(Z-\mu_Z)^2]$$

$$= E[((X+Y) - (\mu_X + \mu_Y))^2]$$

$$= E[((X-\mu_X) + (Y-\mu_Y))^2]$$

$$= E[(X-\mu_X)^2 + 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2]$$

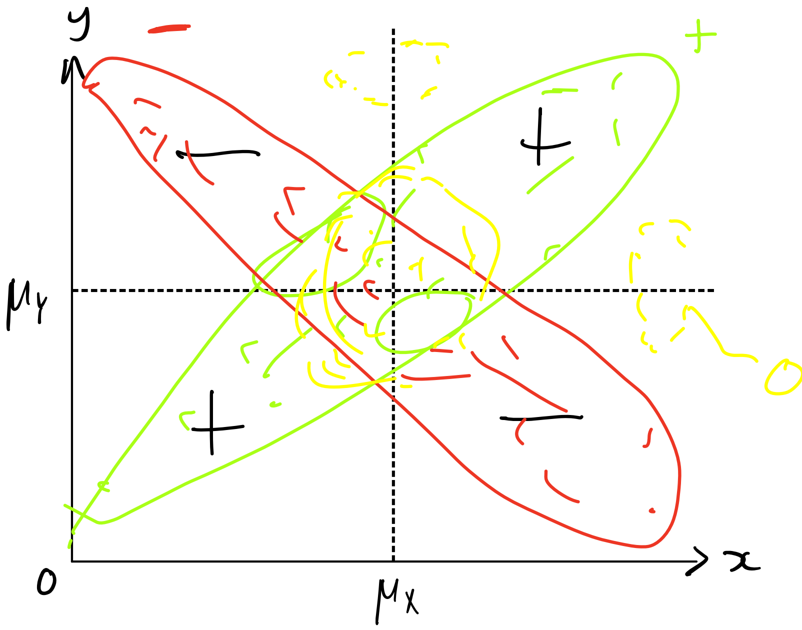
$$= Var[X] + Var[Y] + 2\underbrace{E[(X-\mu_X)(Y-\mu_Y)]}_{Cov[X,Y]}$$

Note:

$$Var[X] = \sigma_X^2$$

$$Cov[X,Y] = \sigma_{X,Y} = \sigma_{XY}$$

$$Cov[X,X] = \sigma_{XX} = \sigma_X^2$$

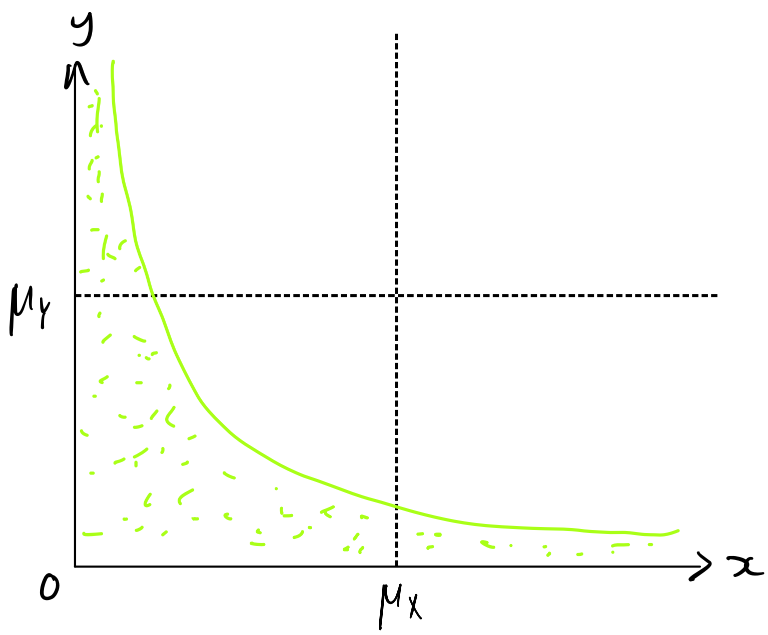


$$Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

$$Cov[X,Y] > 0$$

$$Cov[X,Y] < 0$$

$$Cov[X,Y] \approx 0$$



$>$ or $<$?

$$Cov[X,Y] > 0$$

Example

$$\hat{X} = \frac{X}{100} \qquad \hat{Y} = \frac{Y}{100} \qquad \hat{\mu}_X = \frac{\mu_X}{100} \qquad \hat{\mu}_Y = \frac{\mu_Y}{100}$$

$$\begin{aligned}
Cov[\hat{X}, \hat{Y}] &= E[(\hat{X} - \mu_{\hat{X}})(\hat{Y} - \mu_{\hat{Y}})] \\
&= E\left[\left(\frac{X}{100} - \frac{\mu_X}{100}\right)\left(\frac{Y}{100} - \frac{\mu_Y}{100}\right)\right] \\
&= \frac{E[(X - \mu_X)(Y - \mu_Y)]}{100 \cdot 100} \\
&= \frac{Cov[X, Y]}{10\ 000}
\end{aligned}$$

$$Var[aX] = a^2 Var[X]$$

$$E[XY] = E[X] \cdot E[Y]$$

Examples ρ_{XY}

• X = student height	Y = student weight	$0{,}5 < \rho_{XY} < 1$
• X = distance to cell tower	Y = signal strength	$-1 < \rho_{XY} < 0{,}5$
• X = temp. in °C	Y = temp. in K	$\rho_{XY} = 1$
• X = student number	Y = student cell num.	$\rho_{XY} \approx 0$

$$\begin{aligned}
Cov[X, Y] &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y] \\
&= E[XY] - \mu_Y E[X] - \mu_X E[Y] + \mu_X\mu_Y \\
&= r_{XY} - \mu_X\mu_Y
\end{aligned}$$

Example 5.17

$$r_{X,Y} = E[XY] = \sum_{x \in S_X} \sum_{y \in S_Y} x \cdot y \, P_{X,Y}(x, y)$$

$$= 1 \cdot 1 \cdot 0{,}09 + 2 \cdot 2 \cdot 0{,}81$$

$$= 3{,}33$$

$$Cov[X, Y] = r_{XY} - \mu_X \mu_Y$$

$$= 0{,}252.$$

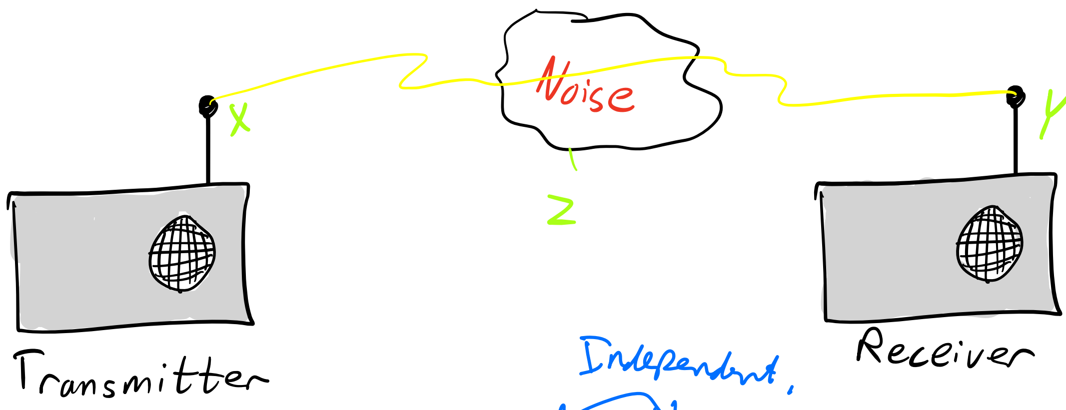
$$\rho_{XY} = \frac{Cov[X, Y]}{\sigma_X \sigma_Y}$$

Strong/
weak.

$\mu_X = E[X] = 1{,}8$

$\mu_Y = E[Y] = 1{,}71$

Example 5.18



Independent,

$$Y = X + Z$$

$$X = \text{Gaus}(0, \sigma_X)$$
$$Z = \text{Gaus}(0, \sigma_Z)$$

$$\rho_{X,Z} = \frac{Cov[X,Z]}{\sqrt{Var[X]\,Var[Z]}} = 0$$

$$\sigma_Y^2 = Var[Y] = Var[X+Z] = Var[X] + Var[Z] = \sigma_X^2 + \sigma_Z^2$$

$$\sigma_Y = \sqrt{\sigma_X^2 + \sigma_Z^2}$$

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]\,Var[Y]}} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} = \frac{\sigma_X^2}{\sqrt{\sigma_X^2(\sigma_X^2+\sigma_Z^2)}}$$

$$Cov[X,Y] = r_{XY} - \mu_X \mu_Y = r_{XY}$$

$$r_{XY} = E[XY] = E[X(X+Z)] = E[X^2 + XZ]$$

$$= E[X^2] + E[XZ]$$

$$= E[X^2] + \cancelto{0}{E[X] \cdot E[Z]}$$

$$Cov[X, Y] = E[X^2] \left(- \underbrace{E[X]^2} \right)$$

$$= Var[X]$$

$$= \sqrt{\sigma^2}$$