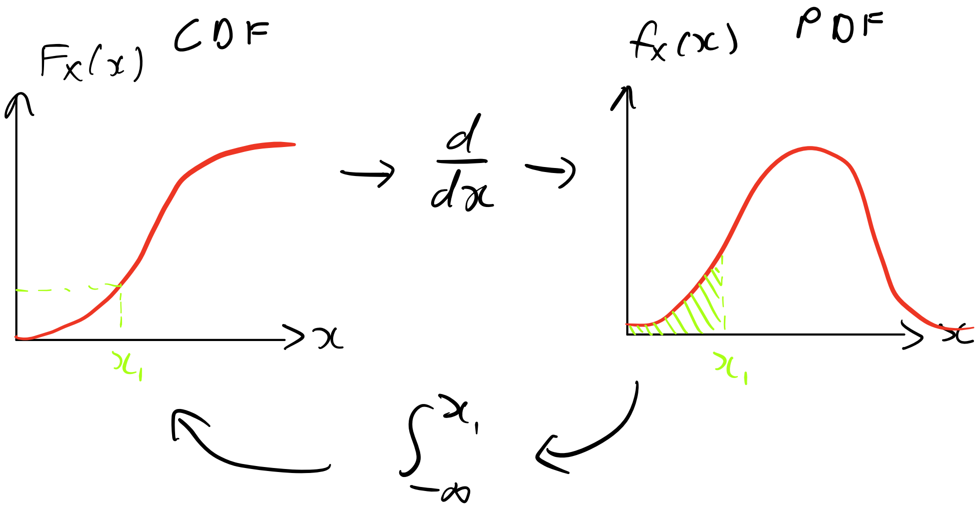
Joint PDFs

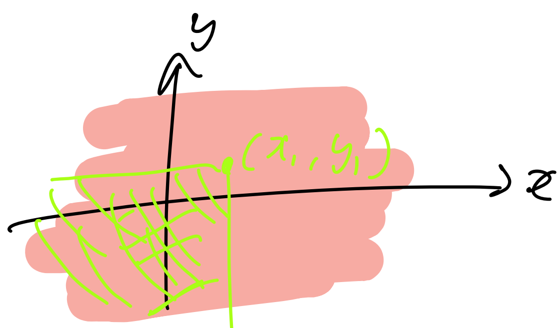
1 Random Variable:

$$f_X(x) = \frac{dF_X(x)}{dx}$$ (slope of CDF)



2 Random Variables

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$



$$F_{X,Y}(x_1,y_1) = \int_{-\infty}^{x_1} \int_{-\infty}^{y_1} f_{X,Y}(x,y)\, dy\, dx$$

Can swap

Example 5.2 → Get Joint PDF $f_{X,Y}(x,y)$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial F_{X,Y}(x,y)}{\partial y} \right]$$

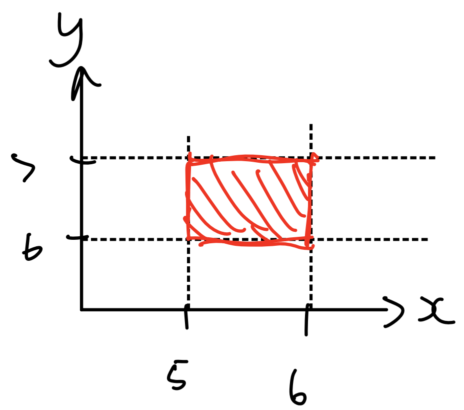
$$= \frac{\partial^2}{\partial x \, dy}\left[(x-5)(y-5)\right]$$

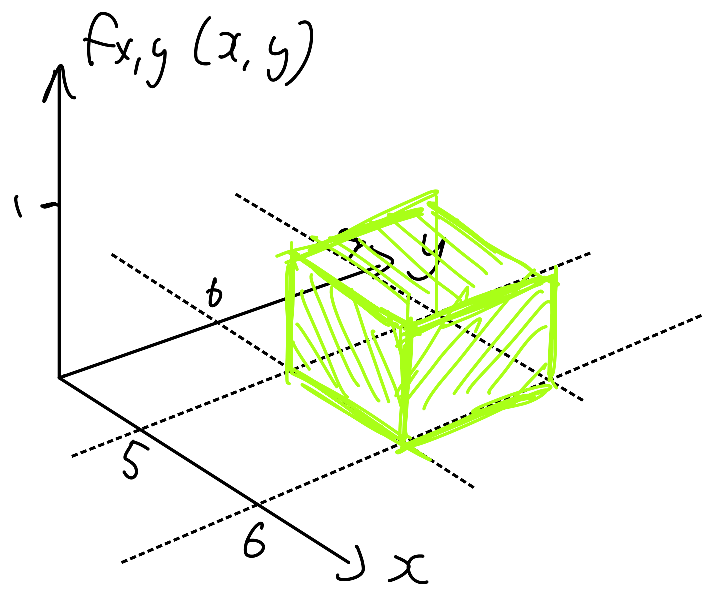
$$= \frac{\partial}{\partial x}(x-5) \cdot \frac{\partial}{\partial y}(y-5)$$

$$1 \quad \cdot \quad 1$$

$$= \begin{cases} 1 & 5 \le x < 6, \; 6 \le y < 7 \\ 0 & \text{otherwise} \end{cases}$$

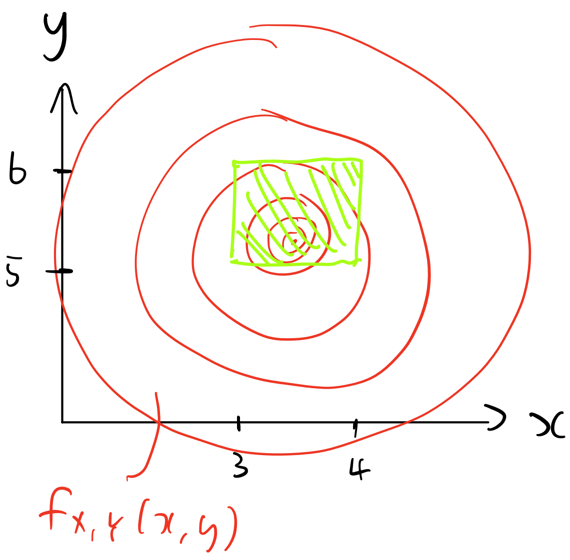
$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)\, dx \, dy.$$





$$P(A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

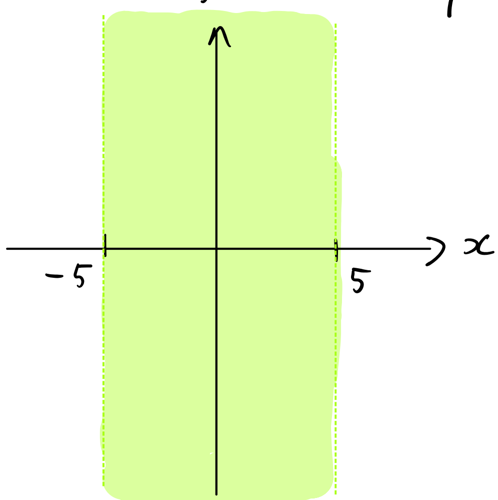
$$A: \left\{ \begin{array}{l} 3 \le x \le 4 \\ 5 \le y \le 6. \end{array} \right\}$$



$$= \int_5^6 \int_3^4 f_{X,Y}(x,y)\,dx\,dy.$$

Marginalization

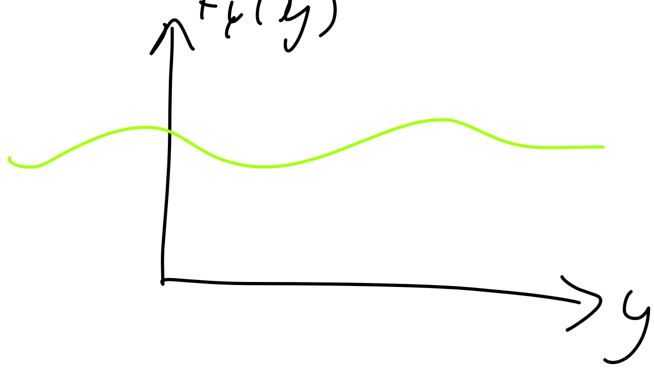
$$F_Y(y_1) = P(Y \le y_1) = \int_{-\infty}^{y_1} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy.$$



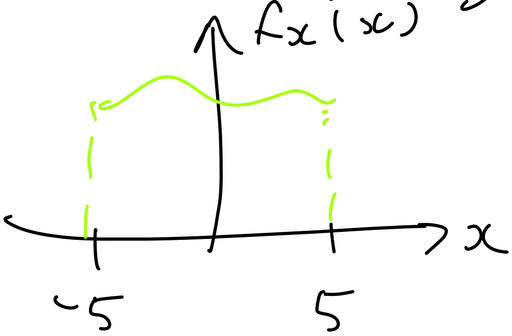
PDF

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$$

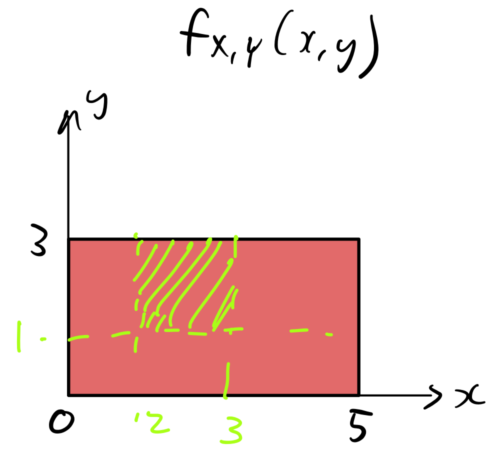
$$= \int_{-5}^{5} f_{X,Y}(x,y)\,dx$$



$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$



Example 5.7



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\int_0^3\int_0^5 C \quad dx\,dy.$$

$$C = \frac{1}{15}$$

$$W \cdot b \cdot h = 1$$

$$5 \cdot 3 \cdot C = 1$$

$$C = \frac{1}{15}$$

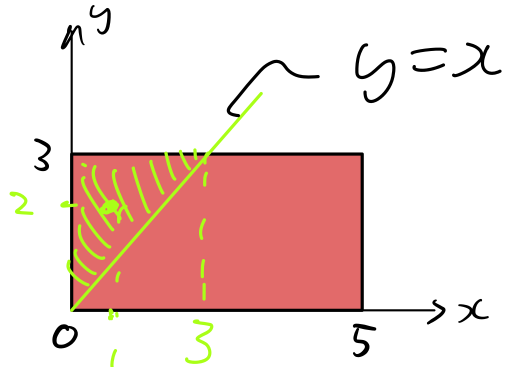
$$P(A) = \int_1^3\int_2^3 \frac{1}{15}\,dx\,dy.$$

$$= W \cdot b \cdot h.$$

$$= 1 \cdot 2 \cdot \frac{1}{15} = \frac{2}{15}$$

Example 5.9

$f_{X,Y}(x,y)$



$$P(A) = P(Y > X)$$

$$= \int_0^3 \int_0^y \frac{1}{15}\, dx\, dy.$$

$$= \int_0^3 \frac{1}{15} x \Big|_0^y \, dy.$$

$$= \int_0^3 \frac{y}{15}\, dy.$$

$$= \frac{1}{2} \cdot \frac{y^2}{15} \Big|_0^3$$

$$= \frac{9}{30} = \frac{3}{10}$$

$$\frac{1}{2} \cdot b \cdot h \cdot H$$

$$\frac{1}{2} \cdot 3 \cdot 3 \cdot \frac{1}{15} = \frac{9}{30}$$

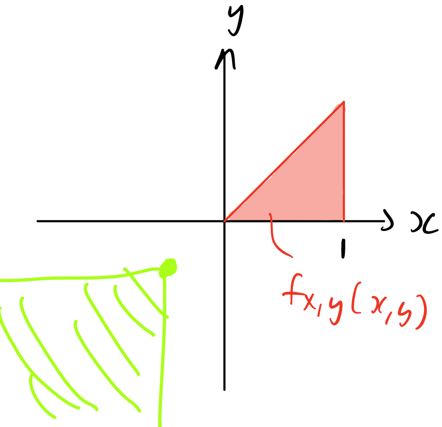
$$= \frac{3}{10}$$

$$F_{X,Y}(x_1, y_1) = \int_{-\infty}^{y_1} \int_{-\infty}^{x_1} f_{XY}(x,y)\, dx\, dy.$$

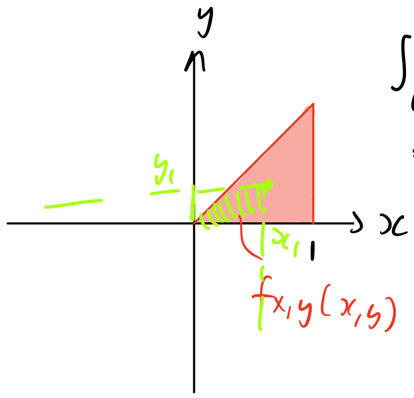
(x, y)

Example 5.8

a) $x < 0$ or $y < 0$

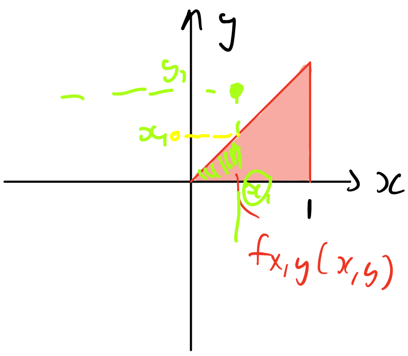


b) $0 \le y \le x \le 1$



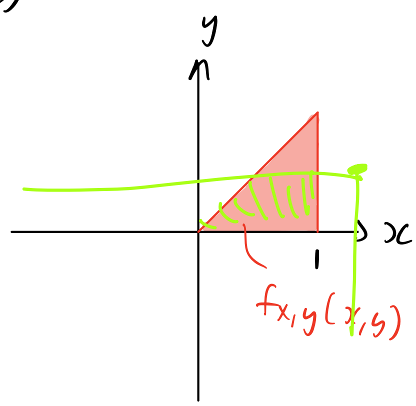
$$\int_0^{y_1}\int_y^{x_1} 2 \, dx \, dy$$
$$= \int_0^{y_1} 2(x_1 - y) \, dy$$
$$= 2x_1 y - y^2 \Big|_0^{y_1}$$
$$= 2x_1 y_1 - y_1^2$$

c) $0 \le x < y$, $0 \le x < 1$



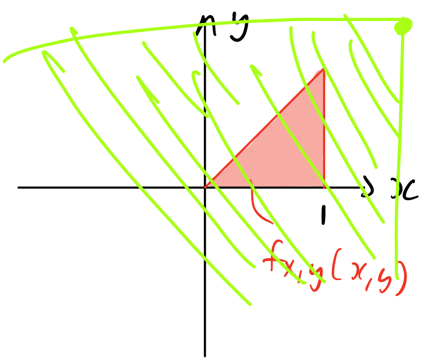
$$\int_0^{x_1}\int_y^{x_1} 2 \, dx \, dy$$
$$\int_0^{x_1} 2x_1 - 2y \, dy$$
$$2x_1 y - y^2 \Big|_0^{x_1}$$
$$= 2x_1^2 - x_1^2$$
$$= x_1^2$$

d) $x > 1$, $0 \le y \le 1$



$$\int_0^{y_1}\int_y^{1} 2 \, dx \, dy$$
$$= 2y - y^2$$

e) $x > 1$, $y > 1$



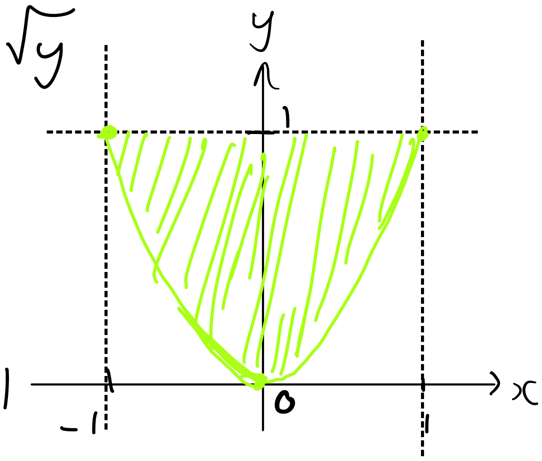
$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \quad (a) \\ 2xy - y^2 & 0 \le y \le x \le 1 \quad (b) \\ x^2 & 0 \le x < y, \ 0 \le x < 1 \quad (c) \\ 2y - y^2 & 0 \le y \le 1, \ x > 1 \quad (d) \\ 1 & x > 1, \ y > 1 \quad (e) \end{cases}$$

Example 5.10

$y = x^2$

$x = \pm\sqrt{y}$

$f_{X,Y}(x,y)$

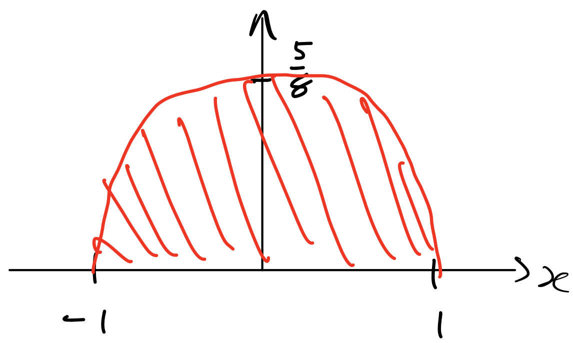


$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{x^2}^{1} \frac{5y}{4}\,dy$$

$$= \begin{cases} \dfrac{5(1-x^4)}{8} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$f_X(x)$



$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{5y}{4}\,dx$$

$$= 2 \cdot \int_{0}^{\sqrt{y}} \frac{5y}{4}\,dx$$

$$= \cancel{2}\, \frac{5}{\cancel{4}}\, y \cdot x \Big|_{0}^{\sqrt{y}} \qquad y^{\frac{1}{2}} = y^{0.5}$$

$$= \frac{5}{2}\, y \cdot \sqrt{y}$$

$$= \begin{cases} \dfrac{5}{2}\, y^{1.5} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$f_Y(y)$