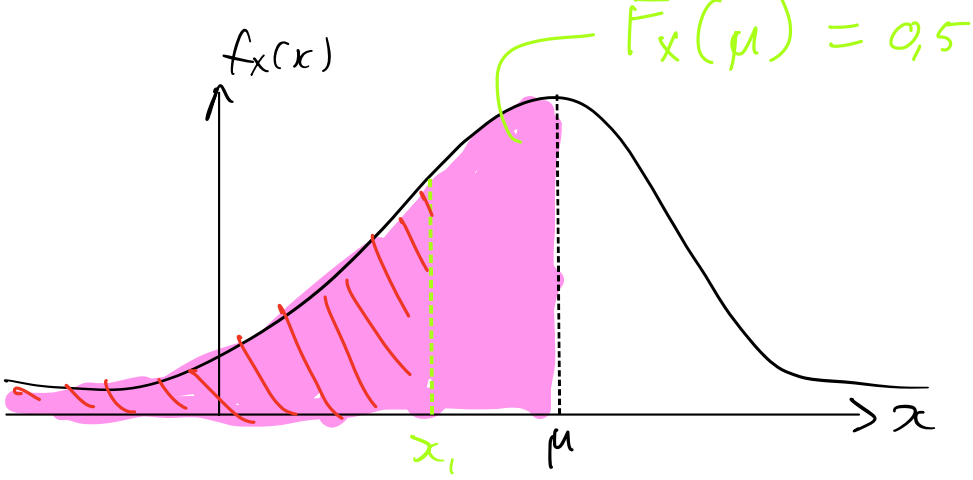
Gaussian Distribution

Gaussian Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

CDF?



$$F_X(x_1) = \int_{-\infty}^{x_1} f_X(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x_1} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Standard Normal Distr. ($\mu=0 \quad \sigma=1 \quad x=z$)

PDF:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = \varphi(z) \sim \text{PDF} \qquad \text{CDF } \Phi(z)$$

To transform any X to Z

Define: $\boxed{Z = g(x) = \frac{x-\mu_X}{\sigma_X}} = \underbrace{\frac{1}{\sigma_X}}_{a\text{ stretch}} x - \underbrace{\frac{\mu_X}{\sigma_X}}_{b\text{ shift}} \quad (ax+b)$

$$\Rightarrow E[z] = \frac{1}{\sigma_X}\mu_X - \frac{\mu_X}{\sigma_X} = 0 = \mu_z$$

$$\Rightarrow Var[z] = \left(\frac{1}{\sigma_X}\right)^2 Var[X]$$

$$= \left(\frac{1}{\sigma_X}\right)^2 \sigma_X^2$$

$$= 1 \qquad \sigma_z = \sqrt{1} = 1$$

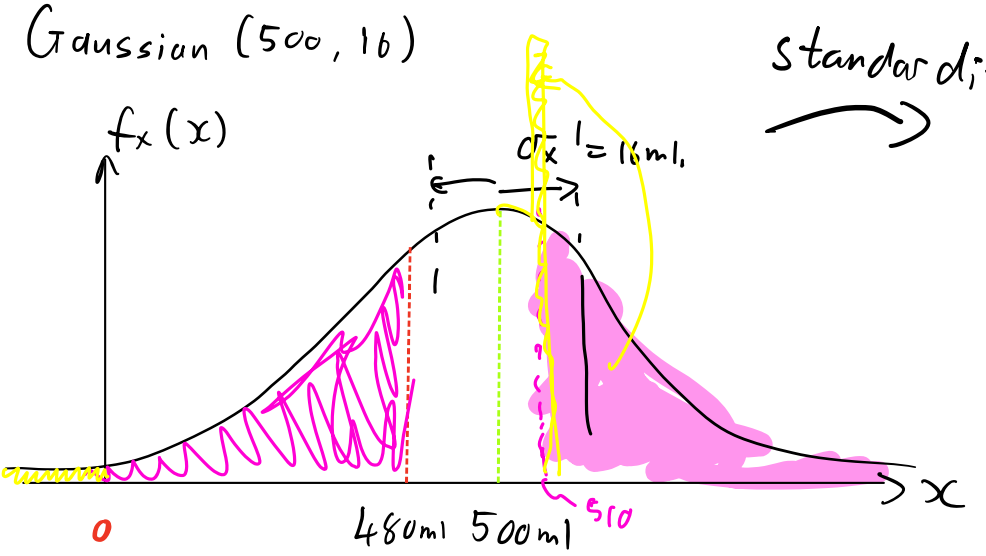
$$\Rightarrow x = g^{-1}(x) = \sigma_X z + \mu_X$$

CDF:

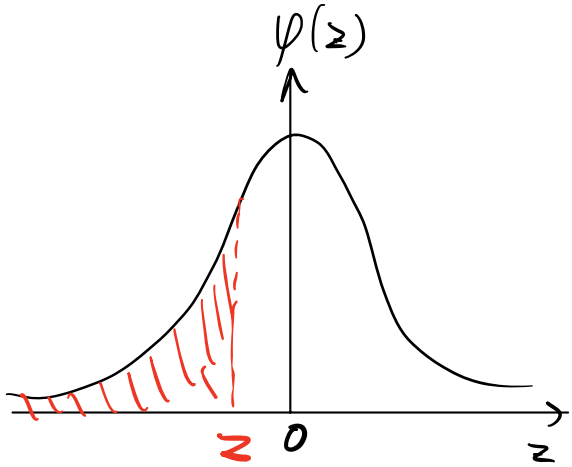
$$F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} du = \Phi(z)$$

Example 1.

Gaussian (500, 16)



standardize



$$P(X \leq 480) = F_X(480)$$

$$= \Phi(-1,25)$$

$$= 1 - \Phi(1,25)$$

$$= 1 - 0,8944$$

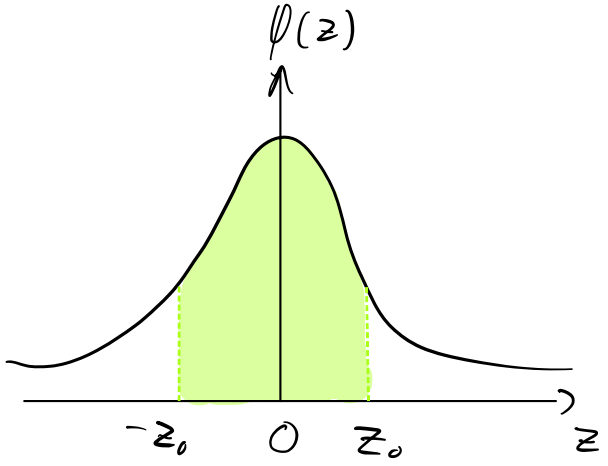
$$= 0,1056 \approx 10,6\%$$

$$z = \frac{x - \mu_X}{\sigma_X}$$

$$= \frac{480 - 500}{16}$$

$$= -1,25$$

Example 2 $\mu_z = 0$ $\sigma_z = 1$



- $P(-z_0 < z \leq z_0)$

$= \Phi(z_0) - \Phi(-z_0)$

$= \Phi(z_0) - (1 - \Phi(z_0))$

$= 2\Phi(z_0) - 1$

- $\sigma_z = 1$ $z_0 = 1$

$= 2\Phi(1) - 1$

$= 2(0,84134) - 1$

$= 0,6826 \approx 68,3\%$ — within 1 stdev

- $2\Phi(z_0) - 1 = 0,99$

$\Phi(z_0) = \frac{1,99}{2} = 0,995$

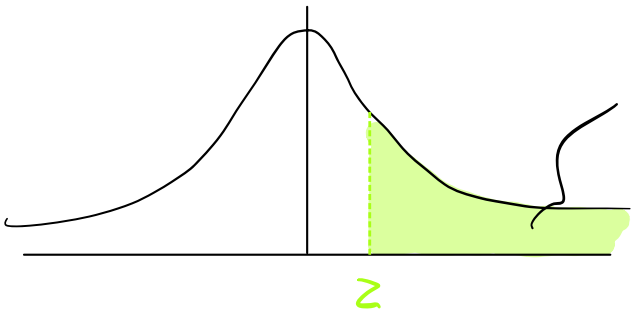
$z_0 = 2,58$

Example 4.12

$x = 46 \qquad \mu = 61 \qquad \sigma = 10$

$$z = \frac{x - \mu}{\sigma} = \frac{46 - 61}{10} = -1,5$$

Complementary



$$P(Z > z) = Q(z) = 1 - \Phi(z)$$

Example 4.14

Bit error rate = Probability of getting error

$$Q(z) < 10^{-6}$$

$$z > 4.75$$

$$\sqrt{\frac{\gamma}{2}} > 4.75$$

$$\gamma > 45 \text{ dB}$$