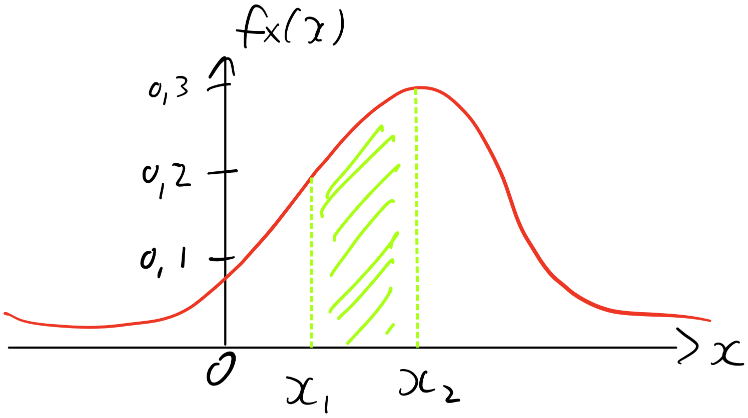
Continuous Random Variables

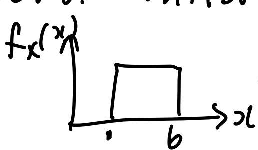


$P(X=x)=0$

$K_2 = \lfloor x \rfloor \rightarrow$ Disc Uni $(1, 5)$

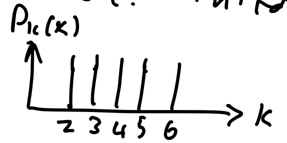
Ceiling $\lceil x \rceil = n$ for $n-1 < x \le n$

Cont. Uniform $(1, 6)$ $\rightarrow$ $\boxed{K = \lceil X \rceil}$ $\rightarrow$ Disc. Uniform $(2, 6)$

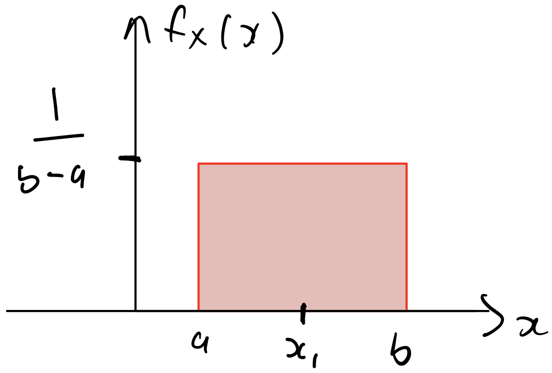


$S_X = [1, 6]$

$P(X=1)=0$



Uniform Distribution



$S_X = [a, b)$ $\qquad$ $S_X = (a, b)$

$P(X = a) = 0$

w. h. $= (b-a) \cdot \boxed{?} = 1$ $\qquad$ $\boxed{?} = \frac{1}{b-a}$

$a \le x_1 < b$:

$$F_X(x_1) = \int_{-\infty}^{x_1} f_X(x)\,dx$$

$$= \int_a^{x_1} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\, x \Big|_a^{x_1} = \frac{x_1 - a}{b-a}$$

General.

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & x \ge b \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$= \int_a^b \frac{x}{b-a}\,dx$$

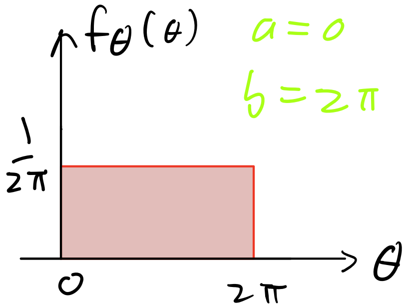
$$= \frac{1}{b-a}\left[\frac{1}{2} x^2 \Big|_a^b\right] = \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

$$= \frac{b+a}{2}$$

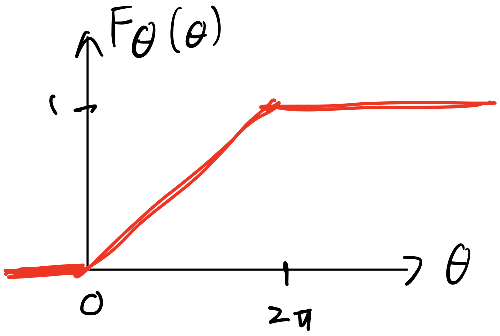
$$Var[X] = E[X^2] - E[X]^2$$

Example 4.5



$a = 0$
$b = 2\pi$

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$$



$$F_\theta(\theta) = \begin{cases} 0 & \theta < 0 \\ \frac{\theta}{2\pi} & 0 \le \theta < 2\pi \\ 1 & \theta \ge 2\pi \end{cases}$$

$$E[\theta] = \frac{2\pi}{2} = 2$$

$$Var[\theta] = \frac{(b-a)^2}{12} = \frac{(2\pi)^2}{12} = \frac{\pi^2}{3}$$

Exponential Distribution

Discrete

How many trials until first success?

↳ Geometric.

Continuous

How long should I wait until event?

↳ Exponential.

Exponential Distribution CDF

$$F_X(x) = \int_{-\infty}^{x} f_V(v)\,dv \qquad V \geq 0$$

$$= \int_0^x \lambda e^{-\lambda v}\,dv$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$= \lambda \left[-\frac{1}{\lambda} e^{-\lambda v} \Big|_0^x \right]$$

$$= -\frac{\cancel{\lambda}^{1}}{\cancel{\lambda}} \left[e^{-\lambda x} - 1 \right]$$

$$= 1 - e^{-\lambda x} \quad \rightarrow \quad x \geq 0$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

$$\boxed{\int e^{ax}\,dx = \frac{1}{a}e^{ax}}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx \qquad x \geq 0$$

$$= \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$$

Integration by Parts

$$\int u\, dv = uv - \int v\, du$$

$$= \lambda \int_0^{\infty} \underbrace{x}_{u} \cdot \underbrace{e^{-\lambda x} dx}_{dv}$$

$$= \lambda \left[x \cdot \left(-\frac{1}{\lambda} e^{-\lambda x}\right)\Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{\lambda} e^{-\lambda x}\, dx \right] \qquad v = -\frac{1}{\lambda} e^{-\lambda x}$$

$$= -\cancel{\frac{\lambda}{\lambda}}^{1} \left[x \cdot e^{-\lambda x}\Big|_0^{\infty} - \left(-\frac{1}{\lambda}\right) e^{-\lambda x}\Big|_0^{\infty} \right]$$

$$= -\left[\underbrace{\left(x + \frac{1}{\lambda}\right)}_{\to\infty} \underbrace{e^{-\lambda x}}_{\to 0} \right]_0^{\infty}$$

L'Hôpitals rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$= -\left[\frac{x + \frac{1}{\lambda}}{e^{\lambda x}} \right]_0^{\infty}$$

$$\lim_{x\to\infty} \left(\frac{1}{\lambda e^{\lambda x}}\right) = 0$$

$$= -\left[0 - \frac{\frac{1}{\lambda}}{1} \right]$$

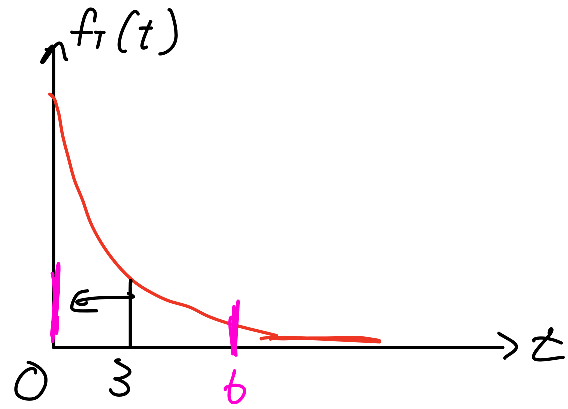
$$= \frac{1}{\lambda}$$

$$\int x e^{ax}\, dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax}$$

Example 4.10

$$\lambda = \frac{1}{3}$$

$$f_T(t) = \begin{cases} \frac{1}{3} e^{-\frac{1}{3}t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$E[T] = \frac{1}{\lambda} = 3 \text{ min.}$$

$$\sigma_T = \frac{1}{\lambda} = 3 \text{ min.}$$

$$P(0 \leq T < 6) = F_T(6) - F_T(0)$$

$$= \left(1 - e^{-\frac{1}{3}\cdot 6}\right) - \overset{1}{\cancel{\left(1 - e^{-\frac{1}{3}\cdot 0}\right)}}$$

$$= 86.5\%$$

Erlang to Geometric or Pascal